On the double cover of split $F_4$
and its petite $K$-types

Alessandra Pantano
joint work with Dan Barbasch

Palo Alto, July 2006
Plan of the talk

• the double cover of split $F_4$

• the big unitarity problem (find all unitary parameters)

• the petit unitarity problem (find some not-unitary parameters)

• an informal definition of non-spherical petite $K$-types

• a formal definition of non-spherical petite $K$-types

• applications to the unitary dual of the double cover of split $F_4$
Plan of the talk

• **the double cover of split $F_4$**

• the big unitarity problem

• the petit unitarity problem

• an informal definition of non-spherical petite $K$-types

• a more technical definition of petite $K$-types

• applications to the unitary dual of the double cover of split $F_4$
The double cover of $F_4$

- $G = \text{the double cover of the split } F_4 \ (F_4 = G/\{\pm I\})$
- $\pi: G \to F_4 = G/\{\pm I\}$, the projection
- $K = SP(1) \times SP(3)$
- Representations of $K$ (classified by highest weight):
  $\mu = (a_1|a_2, a_3, a_4)$, with $a_1 \geq 0$ and $a_2 \geq a_3 \geq a_4 \geq 0$
- Genuine $K$-types ($-I$ does not act trivially):
  $\mu = (a_1|a_2, a_3, a_4)$, with $a_1 + a_2 + a_3 + a_4$ odd
- $g = k \oplus p$: Cartan decomposition of $g$
- $a$: maximal abelian subspace of $p$, $A = \exp(a)$, $M = Z_K(a)$
- $\Delta^+ = \{2\epsilon_j; \epsilon_i \pm \epsilon_j; \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4\}$, $n = \bigoplus_{\alpha \in \Delta^+} g_\alpha$, $N = \exp(n)$
Notations

For each root $\alpha$, we can choose a Lie algebra homomorphism

$$\phi_\alpha : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{g}$$

such that

- $Z_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ belongs to $t = \text{Lie}(K)$

- $\sigma_\alpha = \exp(\frac{\pi}{2} Z_\alpha)$ belongs to $M' = N_K(a)$, and

- $m_\alpha = \exp(\pi Z_\alpha)$ belongs to $M = Z_K(a)$. 
Exponentiating $\phi_\alpha$, we obtain group homomorphisms

$$\tilde{\Phi}_\alpha : \tilde{SL}(2, \mathbb{R}) \to G \quad \Phi_\alpha : SL(2, \mathbb{R}) \to G/\pm I = F_4.$$ 

The root $\alpha$ is called metaplectic if $\tilde{\Phi}_\alpha$ does not factor to $SL(2, \mathbb{R})$.

only the long roots are metaplectic

Consequences:

- If $\alpha$ is short, then $m_\alpha$ has order two and is central in $M$
- If $\alpha$ is long, then $m_\alpha$ has order four and $m_\alpha m_\beta = \pm m_\beta m_\alpha$
- If $\alpha$ is short, the eigenvalues of $d\mu(iZ_\alpha)$ are integers $\forall \mu \in \hat{K}$
- If $\alpha$ is long, the eigenvalues of $d\mu(iZ_\alpha)$ are integers if $\mu$ is not genuine, and half-integers if $\mu$ is genuine.
Let $\mu$ be an irreducible representation of $K$. Then

- $\mu$ has level $l$ if $|\gamma| \leq l$, for every eigenvalue $\gamma$ of $d\mu(iZ_\alpha)$ and every root $\alpha$

- $\mu$ is fine if $\mu$ has level 1 (or less)

There are 2 genuine fine $K$-types: $(1|000)$ and $(0|100)$ and 3 non-genuine fine $K$-types: $(2|000)$, $(1|100)$ and $(0|000)$. 
The group $M$ is a finite group of order 32. Because $\pi(M) = M/\{\pm I\}$ is abelian, the irreducible representations of $M$ have dimension one or two.

There are 16 non-genuine linear characters, and 4 genuine two-dimensional irreducible representations.

The Weyl group acts on $\hat{M}$. The restrictions to $M$ of a fine K-type is a single orbit, and every representation of $M$ is contained in a unique fine K-type.

**Definition:** Fix $\delta \in \hat{M}$. A root $\alpha$ is *good* for $\delta$ if $s_\alpha$ stabilizes $\delta$. 
<table>
<thead>
<tr>
<th>orbit</th>
<th>dim.</th>
<th>$W(\delta)$</th>
<th>fine K-type</th>
</tr>
</thead>
<tbody>
<tr>
<td>non-genuine→ $\delta_0$</td>
<td>1</td>
<td>$W(F_4)$</td>
<td>(0</td>
</tr>
<tr>
<td>non-genuine→ $\delta_3$</td>
<td>3 × 1</td>
<td>$W(C_4)$</td>
<td>(2</td>
</tr>
<tr>
<td>non-genuine→ $\delta_{12}$</td>
<td>12 × 1</td>
<td>$W(B_3A_1)$</td>
<td>(1</td>
</tr>
<tr>
<td>genuine    → $\delta_2$</td>
<td>2</td>
<td>$W(F_4)$</td>
<td>(1</td>
</tr>
<tr>
<td>genuine    → $\delta_6$</td>
<td>3 × 2</td>
<td>$W(B_4)$</td>
<td>(0</td>
</tr>
</tbody>
</table>
Plan of the talk

• the double cover of split $F_4$

• the big unitarity problem

• the petit unitarity problem

• an informal definition of petite $K$-types

• a more technical definition of petite $K$-types

• applications to the unitary dual of the double cover of split $F_4$
For every irreducible representation \((\delta, V^\delta)\) of \(M\), and every strictly dominant real character \(\nu\), we set

\[ X_P(\delta, \nu) = \text{the minimal principal series induced from } \delta \otimes \nu \]

\[ \tilde{X}_P(\delta, \nu) = \text{the unique irreducible composition factor of } X_P(\delta, \nu) \text{ which contains the fine } K\text{-type } \mu_\delta \text{ corresponding to } \delta. \]

The Langlands quotient \(\tilde{X}_P(\delta, \nu)\) can be obtained as the quotient of \(X_P(\delta, \nu)\) modulo the Kernel of an intertwining operator

\[ A: X_P(\delta, \nu) \longrightarrow X_{\tilde{P}}(\delta, \nu) \]

where \(\tilde{P}\) is the opposite parabolic.
For every irreducible representation $\delta$ of $M$, compute the set of unitary parameters

$$\{ \nu \in a \cap \mathbb{R} : \nu \text{ is dominant and } \bar{X}_P(\delta, \nu) \text{ is unitary} \}$$

To check the unitarity of $\bar{X}_P(\delta, \nu)$, we need to
1. construct an invariant Hermitian form on $\bar{X}_P(\delta, \nu)$, if possible
2. verify whether this Hermitian form is positive definite.
**Invariant Hermitian forms on \( \tilde{X}_P(\delta, \nu) \)**

The long Weyl group element of \( F_4 \) (\( \omega = -Id \)) carries \( \delta \) into \( \delta \) and \( \nu \) in \( -\nu \). So we can use \( \omega \) to construct an *Hermitian* intertwining operator

\[
A(\omega, \delta, \nu) : X_P(\delta, \nu) \to X_P(\delta, -\nu).
\]

This operator gives a *non degenerate* invariant Hermitian form on the Langlands quotient.\(^a\)

\( \tilde{X}_P(\delta, \nu) \) is unitary if and only if \( A(\omega, \delta, \nu) \) is positive semidefinite.

---

\(^a\)Because \( \tilde{X}_P(\delta, \nu) \) contains only one copy of the fine \( K \)-type \( \mu_\delta \) corresponding to \( \delta \), we can normalize the operator by requiring that it acts trivially \( \mu_\delta \). Then we obtain the *unique* non-degenerate invariant Hermitian form on \( \tilde{X}_P(\delta, \nu) \).
The big unitarity problem is too hard:

Computing the signature of the operator $A(\omega, \delta, \nu)$ is extremely complicated, especially if the $K$-type is very big. Moreover, we should check the signature on infinitely many $K$-types.

Instead, we look at the petit unitarity problem.
Plan of the talk

• the double cover of split $F_4$

• the big unitarity problem

• the petit unitarity problem

• an informal definition of petite $K$-types

• a more technical definition of petite $K$-types

• applications to the unitary dual of the double cover of split $F_4$
the petit unitarity problem

- find finitely many $K$-types (called “petite”) on which it is easy to compute the signature of the intertwining operator
- use these petite $K$-types to rule out big regions of not-unitarity.

\[\text{a}\text{The notion of spherical petite } K\text{-type is due to Vogan and Barbasch. We will present a generalization to the non-spherical case.}\]
Let be $\mu$ a spherical $K$-type, i.e. assume that $\text{Res}_M(\mu)$ contains the trivial representation of $M$.

$\mu$ is called petite if it has level $\leq 3$.

**Remark:** if $\mu$ is a spherical petite $K$-type, then $d\mu(Z_\alpha^2)$ acts on the isotypic component of the trivial representation of $M$ with eigenvalues 0 or $-4$. This condition makes the intertwining operator on $\mu$ “very special”, and relatively easy to compute.
The intertwining operator has a decomposition as a product of operators corresponding to simple reflections. The factor corresponding to $\alpha$ acts by

$$1 \cdot s_{\alpha} \quad \frac{1 - \langle \gamma, \bar{\alpha} \rangle}{1 + \langle \gamma, \bar{\alpha} \rangle} \cdot s_{\alpha}$$
Intertwining operator on spherical petite $K$-types

On a spherical petite $K$-type the intertwining operator behaves exactly like a p-adic operator.

Because the p-adic spherical unitary dual in known, this matching provides non-unitarity certificates.

We obtain an embedding of the real spherical unitary dual into the p-adic spherical unitary dual.
Plan of the talk

- the double cover of split $F_4$

- the big unitarity problem

- the petit unitarity problem

- **an informal definition of non-spherical petite $K$-types**

- a more technical definition of petite $K$-types

- applications to the unitary dual of the double cover of split $F_4$
To every non-trivial representation $\delta$ of $M$, we associate a real linear group $G_0$ (depending on $\delta$).

A $K$-type $\mu$ containing $\delta$ is called "petite for $\delta$" if the non-spherical intertwining operator for $G$ on $\mu$ matches a spherical intertwining operator for $G_0$ on some petite $K_0$-type $\mu_0$.

The spherical unitary dual of $G_0$ is known, and is detected by a finite number of relevant $K_0$-types.

If we can match all the relevant $K_0$-types, then we obtain non-unitarity certificates for Langlands quotients of $G$:

$$\tilde{X}^G(\delta, \nu) \text{ is unitary } \Rightarrow \tilde{X}^{G_0}(\text{triv}, \nu_0) \text{ is unitary.}$$
The Weyl group $W$ of $G$ acts on $\hat{M}$ by

$$(\sigma \cdot \tau)(m) = \tau(\sigma^{-1}m\sigma).$$

Let $W(\delta) \subseteq W$ be the stabilizer of $\delta$.

It turns out that $W(\delta)$ is the Weyl group of some root system $\Delta_0$. $\Delta_0$ has the same rank as $\Delta$, and in general is not a sub-root system.

We define $G_0$ to be

- the real split group with root system $\Delta_0$ if $\delta$ is non-genuine

- the real split group with root system $\hat{\Delta}_0$ if $\delta$ is genuine.

$G_0$ is always linear, and in general is not a subgroup of $G$. 

<table>
<thead>
<tr>
<th>orbit-type</th>
<th>$\Delta_0$</th>
<th>linear group $G_0(\delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>non-genuine</td>
<td>$\delta_0$</td>
<td>$F_4$ $F_4$</td>
</tr>
<tr>
<td>non-genuine</td>
<td>$\delta_3$</td>
<td>$C_4$ $SP(4)$</td>
</tr>
<tr>
<td>non-genuine</td>
<td>$\delta_{12}$</td>
<td>$B_3A_1$ $SO(3,4) \times SL(2)$</td>
</tr>
<tr>
<td>genuine</td>
<td>$\delta_2$</td>
<td>$F_4$ $F_4^\sim$</td>
</tr>
<tr>
<td>genuine</td>
<td>$\delta_6$</td>
<td>$B_4$ $SP(4)$</td>
</tr>
</tbody>
</table>

If we have “enough” petite $K$-types for $\delta$, then we can relate the unitarity of a Langlands quotient of $G$ induced from $\delta$ to the unitarity of a Langlands quotient of $G_0(\delta)$ induced from the trivial.
the spherical $K_0$-type $\mu_0$

Suppose that there exists a spherical $K_0$-type $\mu_0$ s.t.

1. $\mu_0$ has level at most 3
2. as $W(\delta)$-representations

$$\text{Hom}_M(V^\mu, V^\delta) = \text{Hom}_{M_0}(V^{\mu_0}, V^{\delta_0}).$$

Then $\mu$ is petite if and only if the intertwining operator for $G$ on $\mu$ matches an intertwining operator for $G_0$ on $\mu_0$. 
Plan of the talk

• the double cover of split $F_4$

• the big unitarity problem

• the petit unitarity problem

• an informal definition of non-spherical petite $K$-types

• a more technical definition of petite $K$-types

• applications to the unitary dual of the double cover of split $F_4$
Let $\mu$ be a $K$-type containing $\delta$. If $\mu$ is petite, the intertwining operator on $\mu$ should have certain properties (\ldots).

The intertwining operator acts on

$$\text{Hom}_M(V^\mu, V^{\mu_\delta}) = \bigoplus_j \text{Hom}_M(V^\mu, V^{\delta_j})$$

and depends on the eigenvalues of the $d\mu(Z_\alpha^2)$'s ($\alpha$ simple) on the isotypic component in $\mu$ of all the $M$-types $\delta_j$ in the $W$-orbit of $\delta$.\footnote{$\mu_\delta$ is the unique fine $K$-type containing $\delta$. Every $M$-type $\delta_j$ in the $W$-orbit of $\delta$ appears in $\mu_\delta$ with multiplicity one: $\text{Res}_M(\mu_\delta) = \bigoplus_j \delta_j$.}

To define a petite $K$-type for $\delta$, we essentially need to impose some restrictions on the eigenvalues of the various $Z_\alpha^2$'s.
Let $\mu$ be a $K$-type containing $\delta$. If $\mu$ is petite, the intertwining operator on $\mu$ should have certain properties (...).

The intertwining operator acts on

$$\text{Hom}_M(V^\mu, V^{\mu\delta}) = \bigoplus_j \text{Hom}_M(V^\mu, V^{\delta_j})$$

and depends on the eigenvalues of the various $d\mu(Z_\alpha^2)$'s on the isotypic component in $\mu$ of the $W$-orbit of $\delta$.\(^a\)

It is clear that the definition of petite $K$-type must be a restriction on these eigenvalues.

\(^a\) $\mu_\delta$ is the unique fine $K$-type containing $\delta$. Every $M$-type $\delta_j$ in the $W$-orbit of $\delta$ appears in $\mu_\delta$ with multiplicity one: $\text{Res}_M(\mu_\delta) = \bigoplus_j \delta_j$.  

The intertwining operator on $\mu$ has a factorization as a product of operators $R_\mu(s_\alpha, \gamma)$ corresponding to simple reflections.

The action of a single factor $R_\mu(s_\alpha, \gamma)$ does not respect the decomposition

$$\text{Hom}_M(V^\mu, V^{\mu_\delta}) = \bigoplus_j \text{Hom}_M(V^\mu, V^{\delta_j})$$

but preserves the decomposition of $\text{Hom}_M(V^\mu, V^{\mu_\delta})$ in eigenspaces of $d_\mu(Z^2_\alpha)$:

$$\text{Hom}_M(V^\mu, V^{\mu_\delta}) = \bigoplus_{n \in \mathbb{N}/2} E(-n^2).$$

$R_\mu(s_\alpha, \gamma)$ acts on the $(-n^2)$-eigenspace of $d_\mu(Z^2_\alpha)$ by

$$R_\mu(s_\alpha, \gamma)T(v) = c(\alpha, \gamma, n) \mu_\delta(\sigma_\alpha)T(\mu(\sigma_\alpha)^{-1}v)$$

where $c(\alpha, \gamma, n)$ is a scalar and $\mu_\delta(\sigma_\alpha)T(\mu(\sigma_\alpha)^{-1}v)$ is the action of $s_\alpha$ on $\text{Hom}_M(V^\mu, V^{\mu_\delta})$. 
example 1: $d\mu(Z^2_\alpha)$ has even eigenvalues

The operator $R_\mu(s_\alpha, \gamma)$ acts on $\bigoplus_{n \in 2\mathbb{N}} E(-n^2)$ by

\[
\begin{array}{cccc}
E(0) & E(-4) & E(-16) & E(-36) \\
\bullet & \bullet & \bullet & \bullet \\
\downarrow 1 \cdot s_\alpha & \downarrow \frac{1-x}{1+x} \cdot s_\alpha & \downarrow \frac{1-x}{1+x} \frac{3-x}{3+x} \cdot s_\alpha & \downarrow \frac{1-x}{1+x} \frac{3-x}{3+x} \frac{5-x}{5+x} \cdot s_\alpha \\
\bullet & \bullet & \bullet & \bullet \\
E(0) & E(-4) & E(-16) & E(-36) \\
\end{array}
\]

with $x = \langle \gamma, \bar{\alpha} \rangle$. 
example 2: $d\mu(Z_\alpha^2)$ has odd eigenvalues

The operator $R_\mu(s_\alpha, \gamma)$ acts on $[\bigoplus_{n \in 2\mathbb{N}+1} E(-n^2)]$ by

\[
\begin{align*}
E(-1) & \quad E(-9) & \quad E(-25) & \quad E(-49) \\
\bullet & \quad \bullet & \quad \bullet & \quad \bullet \quad \ldots \\
1 \cdot s_\alpha & \quad \frac{2-x}{2+x} \cdot s_\alpha & \quad \frac{2-x}{2+x} \frac{4-x}{4+x} \cdot s_\alpha & \quad \frac{2-x}{2+x} \frac{4-x}{4+x} \frac{6-x}{6+x} \cdot s_\alpha \\
\cdot & \quad \bullet & \quad \bullet & \quad \bullet \quad \ldots \\
E(-1) & \quad E(-9) & \quad E(-25) & \quad E(-49)
\end{align*}
\]

with $x = \langle \gamma, \vec{\alpha} \rangle$. 
example 3: $d\mu(Z_\alpha^2)$ has half-integers eigenvalues

The operator $R_\mu(s_\alpha, \gamma)$ acts on $\bigoplus_{n\in\mathbb{N}+\frac{1}{2}} E(-n^2)$ by

$$
\begin{align*}
E(-\frac{1}{4}) & \quad E(-\frac{9}{4}) & \quad E(-\frac{25}{4}) & \quad E(-\frac{49}{4}) \\
\bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
1 \cdot s_\alpha & \quad \frac{1}{2} - x \cdot s_\alpha & \quad \frac{3}{2} - x \cdot s_\alpha & \quad \frac{5}{2} - x \cdot s_\alpha \\
\frac{1}{2} + x \cdot s_\alpha & \quad \frac{3}{2} + x \cdot s_\alpha & \quad \frac{5}{2} + x \cdot s_\alpha & \quad \frac{7}{2} + x \cdot s_\alpha \\
\bullet & \quad \bullet & \quad \bullet & \quad \bullet
\end{align*}
$$

with $x = \langle \gamma, \check{\alpha} \rangle$. 
If $\mu$ is a petite $K$-type, every factor $R_\mu(s_{\alpha_i}, \gamma_i)$ of the intertwining operator must satisfy some conditions.

These conditions depend on whether the reflection $s_{\alpha_i}$ stabilizes a certain $M$-type $\delta_i$ in the orbit of $\delta$.

- If $\alpha_i$ stabilizes $\delta_i$ (i.e. it is good for $\delta_i$), then $R_\mu(s_{\alpha_i}, \gamma_i)$ should behave as a factor of a petite spherical intertwining operator.
- If $\alpha_i$ does not stabilize $\delta_i$ (i.e. it is bad for $\delta_i$), then $R_\mu(s_{\alpha_i}, \gamma_i)$ should be independent of the parameter $\gamma_i$.

This behavior is equivalent to some eigenvalues-restrictions.

\[^{\text{a}}\]If $\alpha_1, \alpha_2 \ldots \alpha_r$ are the simple reflections involved in the decomposition, we define inductively $\delta_1 = \delta, \delta_2 = s_{\alpha_1}(\delta_1), \ldots, \delta_r = s_{\alpha_{r-1}}(\delta_{r-1})$.  

---

intertwining operator on non-spherical petite $K$-types
restrictions for $\mu$ petite and $\alpha_i$ good for $\delta_i$

Look at the eigenvalues of $d\mu(Z^2_{\alpha_i})$ on the $\delta_i$-isotypic in $\mu$.
If the eigenvalues are of the form $-(2n)^2$, we only allow 0 and $-4$.

If the eigenvalues are of the form $-(\frac{2n+1}{2})^2$, we only allow $-\frac{1}{4}$, $-\frac{9}{4}$.
restrictions for $\mu$ petite and $\alpha_i$ bad for $\delta_i$

Again, look at the eigenvalues of $d\mu(Z_{\alpha_i}^2)$ on the $\delta_i$-isotypic in $\mu$. If the eigenvalues are of the form $-(2n + 1)^2$, we only allow $-1$

\[ E(-1) \quad E(-9) \quad E(-25) \quad E(-49) \]
\[
\downarrow 1 \cdot s_{\alpha_i}
\]
\[ E(-1) \quad E(-9) \quad E(-25) \quad E(-49) \]

If the eigenvalues are of the form $-\left(\frac{2n+1}{2}\right)^2$, we only allow $-\frac{1}{4}$

\[ E\left(-\frac{1}{4}\right) \quad E\left(-\frac{9}{4}\right) \quad E\left(-\frac{25}{4}\right) \quad E\left(-\frac{49}{4}\right) \]
\[
\downarrow 1 \cdot s_{\alpha_i}
\]
\[ E\left(-\frac{1}{4}\right) \quad E\left(-\frac{9}{4}\right) \quad E\left(-\frac{25}{4}\right) \quad E\left(-\frac{49}{4}\right) \]
The Main Theorem

Let $\mu$ be a petite $K$-type for $\delta$, i.e. assume that $\mu$ satisfies the eigenvalues-conditions described above.

Suppose that there exists a spherical $K_0$-type $\mu_0$ s.t.

1. $\mu_0$ has level at most 3
2. as $W(\delta)$-representations

$$\text{Hom}_M(V^\mu, V^\delta) = \text{Hom}_{M_0}(V^{\mu_0}, V^{\delta_0}).$$

Then the intertwining operator for $G$ on $\mu$ matches an intertwining operator for $G_0$ on $\mu_0$. 
A technical remark

Let $\mu$ be a petite $K$-type. The restrictions on the eigenvalues of $d\mu(Z_{\alpha_i}^2)$ are “local” conditions: they are imposed on the isotypic of the various $\delta_i$ in $\mu$, not “globally” on $\mu$.

It follows that, if $\delta$ is non-trivial, we cannot identify a petite $K$-type for $\delta$ just by looking at its level.\(^a\)

Most often, an explicit construction of the $K$-type is required.\(^b\)

This is just one of the many complications that make the non-spherical case so much harder than the spherical one.

\(^a\)If $\delta$ is trivial, every $K$-type of level at most 3 is petite. If $\delta$ is non-trivial, only about a half of the $K$-types of level 3 turns out to be petite.

\(^b\)We have constructed all our petite $K$-types using mathematica.
genuine petite $K$-types and other $K$-types of level $\leq 3$

<table>
<thead>
<tr>
<th>$K$-type</th>
<th>mult. of $\delta_6$</th>
<th>$K$-type</th>
<th>mult. of $\delta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0</td>
<td>1,0,0)$</td>
<td>1</td>
<td>$(1</td>
</tr>
<tr>
<td>$(2</td>
<td>1,0,0)$</td>
<td>3</td>
<td>$(3</td>
</tr>
<tr>
<td>$(1</td>
<td>2,0,0)$</td>
<td>4</td>
<td>$(1</td>
</tr>
<tr>
<td>$(1</td>
<td>1,1,0)$</td>
<td>4</td>
<td>$(1</td>
</tr>
<tr>
<td>$(0</td>
<td>1,1,1)$</td>
<td>1</td>
<td>$(0</td>
</tr>
<tr>
<td>$(2</td>
<td>1,1,1)$</td>
<td>3</td>
<td>$(2</td>
</tr>
<tr>
<td>$(4</td>
<td>1,0,0)$</td>
<td>5</td>
<td>$(5</td>
</tr>
<tr>
<td>$(3</td>
<td>2,0,0)$</td>
<td>8</td>
<td>$(3</td>
</tr>
<tr>
<td>$(3</td>
<td>1,1,0)$</td>
<td>8</td>
<td>$(3</td>
</tr>
<tr>
<td>$(0</td>
<td>3,0,0)$</td>
<td>5</td>
<td>$(0</td>
</tr>
<tr>
<td>$(2</td>
<td>3,0,0)$</td>
<td>8</td>
<td>$(2</td>
</tr>
<tr>
<td>$(0</td>
<td>2,1,0)$</td>
<td>8</td>
<td>$(0</td>
</tr>
<tr>
<td>$(2</td>
<td>2,1,0)$</td>
<td>5</td>
<td>$(2</td>
</tr>
<tr>
<td>$(1</td>
<td>2,1,1)$</td>
<td>8</td>
<td>$(1</td>
</tr>
</tbody>
</table>

36-1
Plan of the talk

• the double cover of split $F_4$

• the big unitarity problem

• the petit unitarity problem

• an informal definition of non-spherical petite $K$-types

• a more technical definition of petite $K$-types

• applications to the unitary dual
Find a good definition of petite $K$-types

For each given $\delta$, find all the petite $K$-types

For each $\mu$ petite, find the representation of the stabilizer of $\delta$ on $\text{Hom}_M(V^\mu, V^\delta)$. Guess $\mu_0$

Verify that the intertwining operators match $\delta_2, \delta_{12}$ $\rightarrow$ $\delta_3, \delta_6$

If you can match all the relevant $K_0$-types, deduce the existence of an inclusion of unitary duals

Otherwise, compute the intertwining operator on some non-petite $K$-types and see what happens
\section*{Example 1: $\delta_2$}

$\delta_2$ is an irreducible genuine representation of $M$.

The stabilizer of $\delta_2$ is the entire Weyl group $W = W(F_4)$. In particular, every root of $F_4$ is good for $\delta_2$. \textit{This is an easy example!}

We ask whether it is possible to realize all the relevant $W(F_4)$-types using petite $K$-types for $\delta_2$. 
The relevant $W(F_4)$-types are: $1_1$, $2_1$, $2_3$, $4_2$, $8_1$ and $9_1$.

<table>
<thead>
<tr>
<th>petite $K$-type</th>
<th>mult. of $\delta_2$</th>
<th>repres. of $W(F_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1</td>
<td>0,0,0)$</td>
<td>1</td>
</tr>
<tr>
<td>$(3</td>
<td>0,0,0)$</td>
<td>2</td>
</tr>
<tr>
<td>$(1</td>
<td>2,0,0)$</td>
<td>9</td>
</tr>
<tr>
<td>$(1</td>
<td>1,1,0)$</td>
<td>2</td>
</tr>
<tr>
<td>$(0</td>
<td>1,1,1)$</td>
<td>4</td>
</tr>
<tr>
<td>$(0</td>
<td>3,0,0)$</td>
<td>4</td>
</tr>
<tr>
<td>$(0</td>
<td>2,1,0)$</td>
<td>8</td>
</tr>
<tr>
<td>$(1</td>
<td>2,1,1)$</td>
<td>10</td>
</tr>
</tbody>
</table>

We match all of them! So there is an inclusion of unitary duals:

$$\bar{X}^G(\delta_2, \nu) \text{ unitary } \Rightarrow \bar{X}^G(\text{triv}, \nu) \text{ unitary.}$$
Choose a set of simple roots for $G$ (type $F_4$):

\[ \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 \quad 2\epsilon_4 \quad \epsilon_3 - \epsilon_4 \quad \epsilon_2 - \epsilon_3 \]

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \]

$\delta_{12}$ contains 12 one-dimensional representations of $M$. For each of them, the stabilizer is $W(B_3 \times A_1)$.

Let $\bar{\delta}_{12}$ be the character in $\delta_{12}$ that admits

\[ 2\epsilon_4 \quad \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 \quad \epsilon_2 + \epsilon_3 \quad \epsilon_2 - \epsilon_3 \]

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \]

as a basis for the good roots.
The following table shows that we can realize all the relevant $W(B_3)$-types and all the relevant $W(A_1)$-types using petite $K$-types for $\bar{\delta}_{12}$:

<table>
<thead>
<tr>
<th>petite $K$-type</th>
<th>mult. of $\delta_{12}$</th>
<th>repres. of $W(B_3 \times A_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1</td>
<td>1,0,0)$</td>
<td>1</td>
</tr>
<tr>
<td>$(0</td>
<td>1,1,0)$</td>
<td>1</td>
</tr>
<tr>
<td>$(3</td>
<td>1,0,0)$</td>
<td>2</td>
</tr>
<tr>
<td>$(2</td>
<td>1,1,0)$</td>
<td>3</td>
</tr>
<tr>
<td>$(2</td>
<td>2,0,0)$</td>
<td>3</td>
</tr>
<tr>
<td>$(0</td>
<td>2,0,0)$</td>
<td>1</td>
</tr>
</tbody>
</table>
Because we can match all the relevant $W(B_3 \times A_1)$-types, there exists an inclusion of unitary duals:\(^a\)

\[
\begin{align*}
&\bar{X}^G(\delta_{12}, \gamma) \text{ unitary} \Rightarrow \bar{X}^{SO(3,4) \times SL(2)}(\text{triv}, \gamma_0) \text{ unitary} \\
&\text{Notice that there is a shifting of parameters: if } \gamma = (n_1, n_2, n_3, n_4), \\
&\text{then } \gamma_0 = (n_1 + n_4, n_1 - n_4, n_2 + n_3, n_2 - n_3).
\end{align*}
\]

\(^aSO(3, 2) \times SL(2)\) is the real split group with root system $B_3 \times A_1$. 
If $\gamma = (n_1, n_2, n_3, n_4)$ is the parameter for $F_4$, let $\gamma_0 = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4)$ be the corresponding parameter for $B_3 \times A_1$.

The inner product of $\gamma$ with a basis for the good co-roots in $F_4$ should match the inner product of $\gamma_0$ with the simple co-roots in $B_3 \times A_1$:
example 3: $\delta_6$

$\delta_6$ contains three 2-dimensional irreducible representations of $M$. For each of them, the stabilizer of $\delta$ is $W(B_4)$.

Let $\bar{\delta}_6$ the irreducible component of $\delta_6$ that admits

$$2\epsilon_2 \quad \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 \quad 2\epsilon_4 \quad \epsilon_3 - \epsilon_4$$

as a basis for the good roots.

We would like to realize all the relevant $W(B_4)$-types using petite $K$-types for $\bar{\delta}_6$. 
The following is a *complete* list of petite $K$-types for $\tilde{\delta}_6$:

<table>
<thead>
<tr>
<th>petite $K$-type</th>
<th>mult. of $\tilde{\delta}_6$</th>
<th>repres. of $W(B_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0</td>
<td>1,0,0)$</td>
<td>1</td>
</tr>
<tr>
<td>$(2</td>
<td>1,0,0)$</td>
<td>3</td>
</tr>
<tr>
<td>$(1</td>
<td>2,0,0)$</td>
<td>4</td>
</tr>
<tr>
<td>$(1</td>
<td>1,1,0)$</td>
<td>4</td>
</tr>
<tr>
<td>$(0</td>
<td>1,1,1)$</td>
<td>1</td>
</tr>
<tr>
<td>$(2</td>
<td>1,1,1)$</td>
<td>3</td>
</tr>
</tbody>
</table>

The relevant $W(B_4)$-types are:

$$4 \times 0 \quad 31 \times 0 \quad 3 \times 1 \quad \boxed{2 \times 2} \quad 1 \times 3 \quad 0 \times 4.$$ 

**We cannot match** $2 \times 2$!!!
The relevant $W(B_4)$-type $2 \times 2$ is missing. So we cannot deduce an inclusion of unitary duals.

We only get a weaker result:\(^a\)

\[
\begin{align*}
\text{set of unitary parameters for } (\bar{\delta}_6, G) & \subseteq \text{set of unitary parameters for } (\text{triv, } SP(4)) \cup \text{non-unitarity region for } (\text{triv, } SP(4)) \text{ ruled out by } 2 \times 2
\end{align*}
\]

The region ruled out by $2 \times 2$ consists of all parameters of the form $\gamma_0 = (a + 1/2, a - 1/2, b, 1)$ with $(a, b)$ in the triangle delimited by the lines $a = 1/2$, $b = 0$ and $a + b = 3/2$.

\(^a\)Notice that the stabilizer of $\tilde{\delta}_6$ is of type $B_4$ but we are taking $G_0 = SP(4)$. Indeed, $\tilde{\delta}_6$ is genuine, so $G_0$ must be the split group with co-roots of type $B_4$. 
example 4: $\delta_3$

$\delta_3$ contains three 1-dimensional irreducible representations of $M$. For each of them, the stabilizer of $\delta$ is $W(C^4)$.

Let $\bar{\delta}_3$ the irreducible component of $\delta_3$ that admits

$$
\begin{align*}
\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 & \quad \epsilon_3 + \epsilon_4 & \quad \epsilon_2 - \epsilon_3 & \quad \epsilon_3 - \epsilon_4
\end{align*}
$$

as a basis for the good roots.

Next, we look at the complete list of petite $K$-types for $\bar{\delta}_3$, and we hope to realize all the relevant $W(C^4)$-types: $4 \times 0 \quad 0 \times 4 \quad 3 \times 1 \quad 1 \times 3 \quad 2 \times 2 \quad 31 \times 0$. 
<table>
<thead>
<tr>
<th>petite $K$-type</th>
<th>mult. of $\bar{\delta}_3$</th>
<th>repres. of $W(C_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2</td>
<td>0,0,0)$</td>
<td>1</td>
</tr>
<tr>
<td>$(4</td>
<td>0,0,0)$</td>
<td>1</td>
</tr>
<tr>
<td>$(0</td>
<td>2,0,0)$</td>
<td>3</td>
</tr>
<tr>
<td>$(2</td>
<td>2,0,0)$</td>
<td>6</td>
</tr>
<tr>
<td>$(2</td>
<td>1,1,0)$</td>
<td>2</td>
</tr>
<tr>
<td>$(1</td>
<td>3,0,0)$</td>
<td>4</td>
</tr>
<tr>
<td>$(1</td>
<td>2,1,0)$</td>
<td>8</td>
</tr>
<tr>
<td>$(1</td>
<td>1,1,1)$</td>
<td>4</td>
</tr>
<tr>
<td>$(0</td>
<td>2,1,1)$</td>
<td>3</td>
</tr>
<tr>
<td>$(2</td>
<td>2,1,1)$</td>
<td>7</td>
</tr>
</tbody>
</table>

We cannot match $1 \times 3$!!!
The relevant $W(C_4)$-type $1 \times 3$ is missing. So we cannot deduce an inclusion of unitary duals.

Just like before, we only obtain a weaker result:

\[
\begin{array}{c}
\text{set of unitary parameters for } (\bar{\delta}_3, G') \\
\subseteq \\
\text{set of unitary parameters for } (\text{triv, } SP(4)) \\
\cup \\
\text{non-unitarity region for } (\text{triv, } SP(4)) \\
\text{ruled out by } 1 \times 3
\end{array}
\]

The region ruled out by $1 \times 3$ is the \textit{line segment}

\[\gamma_0 = (3/2 + t, 1/2 + t, -1/2 + t, -3/2 + t)\]

with $1/2 \leq t \leq 3/2$. 
Understand if these “extra regions” contain any unitarity point.