Abstract. Let \( G \) be the real points of a complex connected reductive algebraic group \( G_C \). Let \( K \) be a maximal compact subgroup of \( G \). We parametrize the set \( \hat{K} \) of irreducible representations of \( K \). The goal is to describe an algorithm for such a parametrization and to implement it as a package of the Atlas of Lie groups and representations software developed by Fokko du Cloux.

1. Introduction

Let \( G_C \) be a complex connected reductive algebraic group and \( G \) the set of real points of \( G_C \). Let \( \theta \) be the Cartan involution of \( G \) which extends to an involution of \( G_C \). We denote by \( K \) a maximal compact subgroup of \( G \). Then \( G_C^\theta = K_C \) the complexification of \( K \). We identify \( G \) with a root datum \((X^*, \Delta^+, X_*, (\Delta^+)^\vee)\). So we would like to describe \( \hat{K} \) in term of \( X^* \).

Let \( H \) be a \( \theta \)-stable Cartan subgroup of \( G \) and \( \Delta(g_C, h_C) \) the corresponding root system of \( g_C = \text{Lie}(G_C) \). \( \Delta_{im} \) and \( \Delta_{re} \) will denote the sets of imaginary and real roots in \( \Delta(g_C, h_C) \) respectively. Then \( X^*(H_C) \) the character lattice of \( H_C \) is isomorphic to \( X^* \). Finally, let \( T = K \cap H \) a compact, possibly disconnected torus.

We have the following lemma:

**Lemma 1.1.** The set of characters of \( T \) is isomorphic to \( \frac{X^*(H_C)}{(1-\theta)X^*(H_C)} \).

Let \( \rho \) be half the sum of positive roots in \( \Delta(g_C, h_C) \) and fix

\[
\lambda \in \frac{X^*(H_C) + \rho}{(1-\theta)X^*(H_C)}
\]

We want \( I(H, \Delta_{im}^+, \Delta_{re}^+, \lambda) \) to correspond to a virtual representation of \( G \) restricted to \( K \). Consider a discrete series representation of \( G \) restricted to \( K \) for example. The main idea is to describe irreducible representations of \( K \) as lowest \( K \)-types.

Let \( 2\rho_r^\vee \) be the sum of positive real coroots. Define

\[
\Delta_T = \{ \text{roots } \perp 2\rho_r^\vee \}.
\]

Key words and phrases.
Then $\Delta_T$ is a $\theta$-stable root system corresponding to a real Levi subgroup $L$ of $G$ with $H$ fundamental in $L$. Fix $\Delta_T^+$ containing $\Delta_{im}^+$ and consider the set

$$\mathcal{L} = \{ \lambda \in \frac{X^*(H_c) + \rho}{(1 - \theta)X^*(H_c)} \}$$

such that

1. $\lambda$ is weakly dominant for $\Delta_T^+$
2. if $\alpha$ is a simple imaginary root and $\langle \lambda, \alpha^\vee \rangle = 0$ then $\alpha$ is non-compact.
3. if $\beta$ is a simple real root then $\langle \lambda, \beta^\vee \rangle$ is odd.

(1) ensures that $I(H, \Delta_{im}^+, \Delta_{re}^+, \lambda)$ is a standard limit representation of $G$ restricted to $K$.

(2) ensures that $I(H, \Delta_{im}^+, \Delta_{re}^+, \lambda)$ is non-zero.

(3) ensures that $I(H, \Delta_{im}^+, \Delta_{re}^+, \lambda)$ cannot be written using a more compact Cartan subgroup and Hecht-Schmid identities.

Given that $\lambda$ is defined modulo $(1 - \theta)X^*(H_c)$ to see that property (3) is well-defined one only needs to consider that for $\gamma \in X^*(H_c)$,

$$\langle \gamma - \theta \gamma, \beta^\vee \rangle = \langle \gamma, \beta^\vee - \theta \beta^\vee \rangle = \langle \gamma, 2 \beta^\vee \rangle$$

which is even.

The main theorem is:

**Theorem 1.2.** If $\lambda \in \mathcal{L}$ then $I(H, \Delta_{im}^+, \Delta_{re}^+, \lambda)$ has a unique lowest $K$-type $\mu(H, \Delta_{im}^+, \Delta_{re}^+, \lambda)$.

Hence

$$\hat{K} = \coprod_{[H \text{ mod conjugation by } K]} \coprod_{\Delta_{im}^+ \text{ mod conjugation by } W(G,H)} \mu(H, \Delta_{im}^+, \Delta_{re}^+, \lambda).$$

To see why conjugation under $\Delta_{re}$ does not interfere with this parametrization one has to observe that if $\lambda \simeq \lambda'$ for $\lambda$ and $\lambda' \in \mathcal{L}$ then for $\beta \in \Delta_{re}$

$$\lambda' = s_\beta(\lambda) - [\rho_R - s_\beta(\rho_R)] = \lambda - (\langle \lambda, \beta^\vee \rangle + 1)\beta = \lambda - 2m\beta \text{ with } m \in \mathbb{Z}$$

But $2m\beta = m\beta - \theta(m\beta) = m(1 - \theta)\beta$.

2. **Algorithm**

To follow

**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, 100 MORRISSEY BOULEVARD, BOSTON, MA 02125-3393**

**E-mail address:** anoel@math.umb.edu