Spherical Unitary Representations of Split Groups

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Abstract

This is an expository version of the first few sections of *Spherical Unitary Dual for Split Classical Groups* by Dan Barbasch. See www.math.cornell.edu/~barbasch.

1 Introduction

Let $G$ be a split symplectic or orthogonal group over $\mathbb{R}$ or a p-adic field. We compute the irreducible unitary spherical representations of $G$.

Suppose $\lambda = (a_1, \ldots, a_n)$ where $n = \text{rank}(G)$, $a_i \in \mathbb{C}$ for all $i$. Then associated to $\lambda$ is a principal series representation $X(\lambda)$. This representation has a unique spherical constituent which we denote $\overline{X}(\lambda)$. This is tempered and hence unitary if $a_i \in i\mathbb{R}$ for all $i$. Unitarity for general $\lambda$ reduces to the case $a_i \in \mathbb{R}$ for all $i$ [?]. From now on we assume this is the case. Then $X(\lambda)$ has an invariant Hermitian form if and only if $-\lambda$ is conjugate to $\lambda$ by the Weyl group.

This is automatic if the long element of the Weyl group is equal to $-1$, i.e. $G$ is not of type $D_n$ with $n$ odd. In the latter case the condition holds if and only if $a_i = 0$ for some $i$.

Let $G^\vee$ be the complex dual group of $G$. Fix a unipotent orbit $O^\vee$ of $G^\vee$. According to the Arthur conjectures [?] associated to $O^\vee$ is (among other things) a spherical unitary representation $\pi$ of $G$. By standard theory attached to $O^\vee$ is semi-simple $\text{Ad}(G^\vee)$ orbit in the Lie algebra $g^\vee$ of $G^\vee$, which in turn gives rise to element $\lambda \in h^*$. We write $\lambda = \lambda(O^\vee)$. This spherical representation associated to $O^\vee$ by Arthur’s conjecture is $\overline{X}(\lambda)$, i.e. we expect that $\overline{X}(\lambda)$ is unitary.

For example the principal nilpotent orbit gives $\lambda = \lambda(O^\vee) = \rho$ and $\overline{X}(\lambda)$ is the trivial representation. On the other hand if $O^\vee = 0$ then $\lambda = \lambda(O_c) = 0$, and $X(\lambda) = \overline{X}(\lambda)$ is irreducible and unitary.

Associated to $O^\vee$ is the Bala–Carter [?] Levi factor $M^\vee$ of $G^\vee$. If $M^\vee = G^\vee$ the orbit $O^\vee$ is said to be distinguished. The Levi factor $M^\vee$ has the property that $O^\vee \cap M^\vee$ is a distinguished nilpotent orbit $O^\vee_M$ in $M^\vee$. Furthermore the split $F$–form $M$ of the dual of $M^\vee$ is then a Levi subgroup of $G$. Suppose $M$ is a
proper subgroup of $G$. By the preceding discussion we expect that the spherical representation $\chi_M(O_M)$ of $M_c$ is unitary.

In a bit more detail, we have

$$M \simeq M_0 \times GL(m_1) \times \ldots \times GL(m_r)$$

where $M_0$ is of the same type as $G$. The only distinguished nilpotent orbit in $GL(m)$ is the principal nilpotent, so $O^\vee$ is the product of a distinguished nilpotent orbit in $M_0$ with the principal nilpotent orbits in each $GL$ factor.

Let us assume for the moment that for any distinguished nilpotent orbit of $M_0$ the corresponding spherical representation $\chi_M$ is unitary.

Now suppose $O_M$ is not distinguished, with corresponding Levi factor $M$ and $O_M^\vee = O^\vee \cap M^\vee$. Let $\lambda = \lambda(O^\vee) = \lambda(O_M^\vee)$. By the preceding discussion we assume $\chi_M(\lambda)$ is unitary. Then $\chi(\lambda)$ is the spherical constituent of

$$Ind_G^P(\chi_M(\lambda) \otimes 1)$$

where $Ind_G^P$ denotes unitary induction from $P = MN$ to $G$. In particular $\chi(\lambda)$ is unitary. Henceforth we drop $N$ from the notation and write $Ind_G^M(\chi_M(\lambda))$.

From this realization of $\chi(\lambda)$ we see it may be possible to embed $\chi(\lambda)$ in a continuous family of unitary representations. Let $\chi$ be a real-valued character $\chi$ of $M$ (trivial on $M_0$), i.e. $\chi$ restricted to each $GL$ factor is a real power of $|det|$. We may then consider $Ind_G^M(\chi_M(\lambda)\chi)$ Letting $\nu = d\chi$ we write this as

$$(1.1) \quad Ind_G^M(\chi_M(\lambda + \nu))$$

and the spherical constituent of this representation is $\chi(\lambda + \nu)$. In fact the induced representation (1.1) is reasonably close to be irreducible. More precisely the multiplicities of certain $K$-types which determine unitarity are the same in (1.1) and $\chi(\lambda + \nu)$.

Suppose $Ind_G^M(\chi_M(\lambda))$ is irreducible. It is well known that the signature of the invariant Hermitian form on $Ind_G^M(\chi_M(\lambda + \nu))$, as $\nu$ varies, can only change sign at a point where it is reducible. We conclude that $\chi(\lambda + \nu)$ is unitary for all $\nu$ in some open set. This is the complementary series attached to $O^\vee$ and containing $\chi(\lambda)$. This complementary series exists for the induced representation 1.1 (even it is not irreducible). We seek to describe this set.

If $O^\vee$ is the 0-orbit then $M \simeq GL(1)^n$ is the split torus in $G$, and $\chi(\nu)$ is the spherical constituent of the minimal principal series representation $Ind_G^M(\nu)$. The 0-complementary series may be considered as an open subset of $\mathbb{R}^n$. We are going to reduce to this case, so we assume that we have computed this set for all classical groups.

We return to the consideration of a general nilpotent orbit $O^\vee$.

**Definition 1.1** Given $O^\vee$ we let $H^\vee$ be the reductive part of the centralizer of $O^\vee$ in $G^\vee$. Let $H$ be the $\mathbb{F}$-points of the split $\mathbb{F}$-form of the dual group of $H^\vee$.

**Remark 1.2** The identity component of $H$ is a product of symplectic and orthogonal groups.
Note that $H$ is not necessarily a subgroup of $G$. By $[?]$, $M^\vee$ is the centralizer in $G^\vee$ of a maximal torus $T^\vee$ in $H^\vee$ and $T^\vee$ is the center of $M^\vee$. (In particular $O^\vee$ is distinguished if and only if $H^\vee$ is finite.) Taking duals we see that the maximal split torus $T$ of $H$ may identified with the center of $M$. Consequently the character $\nu = d\chi$ of $M$ may be identified with a minimal principal series representation $Ind_H^G(\nu)$.

The key observation is that the complementary series containing $X(\lambda)$ is determined by the 0-complementary series of $H$:

**Proposition 1.3** The representation $X_G(\lambda + \nu)$ is unitary if and only if $X_H(\nu)$ is unitary, i.e. $X_H(\nu)$ is in the 0-complementary series for $H$.

We now have a large family of unitary representations obtained by continuous deformation of the representations associated to a nilpotent orbit. The main theorem is that this gives the entire spherical unitary dual.

**Theorem 1.4** Let $G = Sp(n,F)$ or $SO(n,F)$ be a split group over a $F = \mathbb{R}$ or a $p$-adic field.

1. Let $O^\vee$ be a distinguished nilpotent orbit in $G^\vee$, and let $\lambda = \lambda(O^\vee)$. Then $X(\lambda)$ is unitary.

2. Fix a nilpotent orbit $O^\vee$ and let $\lambda = \lambda(O^\vee)$. Let $H = H(O^\vee)$ (Definition 1.1). The complementary series $X(\lambda + \nu)$ associated to $O^\vee$ is in bijection with the 0-complementary series $X_H(\nu)$ of $H$.

3. Suppose $\pi$ is an irreducible unitary spherical representation of $G$. Then there is a unique nilpotent orbit $O^\vee$ such that $\pi \simeq X(\lambda + \nu)$ where $\lambda = \lambda(O^\vee)$ and $X(\lambda + \nu)$ is in the complementary series attached to $O^\vee$.

By Remark 1.2 the next result completes the classification of the spherical unitary dual.

**Theorem 1.5** Classification of 0-complementary series for types $B,C,D$.

By the preceding discussion is an algorithm which associates to any $\lambda$ a group $H$ and a parameter $\nu$ for $H$ such that $X_G(\lambda)$ is unitary if and only if $X_H(\nu)$ is unitary. We make this algorithm explicit in Section 3.

## 2 Data associated to a nilpotent orbit

We describe some data associated to a nilpotent orbit in a classical group. This will be applied to $G^\vee$.

Let $G = Sp(n,C)$ or $SO(n,C)$. The nilpotent orbits of $G$ are parametrized by partitions $(a_1, \ldots, a_r)$ with $a_1 \geq \ldots a_n \geq 0$ and $\sum a_i = n$. The multiplicity of each even (respectively odd) part must be even in the case of $O(n)$ (resp. $Sp(n)$).

We view the partition as a Young diagram with rows of length $a_1, \ldots, a_r$. 

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The parameter $h$: We first give an algorithm to compute $h = h(O)$. For each row of length $a_i > 1$ we attach the set $\{1, 2, \ldots, \frac{a_i - 1}{2}\}$ if $a_i$ is odd, or $\{\frac{a_i}{2}, \ldots, \frac{a_i - 1}{2}\}$ if $a_i$ is even. Let $S$ be the union of these sets and let $h_0$ be the elements of $S$ arranged in decreasing order. Then $h$ is obtained by appending $0$'s to $h_0$ so that the number of coordinates is the rank of $G$.

The group $H$: Fix a partition $P$. We write

$$P = (a_1^{m_1}, a_2^{m_2}, \ldots, a_r^{m_r})$$

where $a^m = a.a.\ldots.a$.

For each $i$, we let

$$H_i = \begin{cases} 
O(m_i) & G = Sp(n), a_i \text{ even} \\
O(m_i) & G = O(n), a_i \text{ odd} \\
Sp(n) & G = Sp(n), a_i \text{ odd} \\
Sp(n) & G = O(n), a_i \text{ even} 
\end{cases}$$

Then

$$H = S[H_1 \times \ldots H_r]$$

Note that $H$ contains a non-trivial torus if and only if $m_i > 1$ for some $i$. Therefore

$O$ is distinguished if and only if all rows have distinct length

A nilpotent is even [?] if all rows have the same parity. If a row of length $a$ multiplicity one then $a$ is even (resp. odd) if $G = Sp(n)$ (resp. $SO(n)$). Therefore all distinguished nilpotent orbits are even.

The group $M$:

Let $P$ be a partition

$$P = (a_1^{m_1}, \ldots, a_r^{m_r})$$

as above, corresponding to a nilpotent orbit $O$ of $G$. We make new partitions $(P_0, P_1)$ as follows. The partition $P_1$ is obtained from $P$ by replacing each odd $m_i$ with $m_i - 1$, and $P_0$ has a single row of length $a_i$ for each odd $m_i$. That is $P_0 \cup P_1 = P$, the multiplicity of each row in $P_1$ is even, and the rows of $P_0$ each have multiplicity one.

Now suppose $P$ corresponds to a nilpotent orbit for $G$. Write

$$P_0 = (a_1, \ldots, a_r)$$

$$P_1 = (b_1^{m_1}, \ldots, b_r^{m_r})$$

Each $m_i$ is even. Let $M_0$ be a classical group of the same type as $G$ and of rank $\sum a_i$. Then

$$M = M_0 \times GL(b_1)^{\frac{m_1}{2}} \times GL(b_r)^{\frac{m_r}{2}}.$$ 

Note that the orbit $O_0$ in $M_0$ corresponding to $P_0$ is distinguished.
3 Algorithm

In this section we give an explicit algorithm realizing Theorem 1.4. That is we show how to decide whether a given representation $\overline{X}(\lambda)$ is unitary.

Fix $\lambda$. To determine if $\overline{X}(\lambda)$ is unitary we need to know if we can write $\overline{X}(\lambda)$ as in Theorem 1.4 (3). We begin with some combinatorial considerations.

Define an equivalence relation $\sim$ on $\mathbb{R}$: $a \sim b$ if $a + b$ or $a - b$ is an integer. The equivalence classes are in bijection with $[0, 1/2]$. If $S$ is a finite subset of $\mathbb{R}$ we write $S$ as a disjoint union of equivalence classes $S_0 \cup S_1 \cup \ldots S_r$. Here we will require $S_0$ is the set of elements of $S$ in $\mathbb{Z}$ or $\mathbb{Z} + 1/2$ depending on the situation.

By a string we mean a set of real numbers of the form $\{a, a-1, \ldots, a-\ell\}$. By a balanced string we mean a string of the form $\{a, a-1, \ldots, -a\}$. Note that this implies $2a \in \mathbb{Z}$.

Fix a set $T = \{b_1, \ldots, b_r\}$ of non-negative real numbers which are all equivalent. We seek to write $T$ as a disjoint union of strings, where we allow each $b_i$ to be replaced by $-b_i$. That is we write

$$T = |T_1| \cup |T_2| \cup \cdots \cup |T_s|$$

each $T_i$ is a string and $|T_i| = \{|b| \mid b \in T_i\}$.

We construct these sets inductively. Assume $b_1 \geq b_2 \ldots b_r \geq 0$.

Let $T_1$ be the maximal string containing $b_1$ made from $b_1, \pm b_2, \ldots, \pm b_r$. That is $T_1 = \{b_1, b_1-1, \ldots, b_1-\ell\}$ where $\ell$ is maximal so that $b_1, \pm b_2, \ldots, \pm b_1-\ell \in T$. Write $T = T_1 \cup (T - T_1)$. Apply the same procedure to $T - T_1$. Proceeding in this way we obtain sets $T_i$ as stated.

We say $T$ is the union of the strings $T_i$. (This is a slight abuse of notation: in fact $T = \cup |T_i|$.)

If each $b_i \in \frac{1}{2}\mathbb{Z}$ we may further require that each string $T_i$ is balanced. We can not necessarily write $T$ as a union of balanced strings. However there is a unique maximal subset which can be so written, and we have

$$T = T' \cup |T_1| \cup \cdots \cup |T_r|$$

where $T_1, \ldots, T_r$ are balanced and $T'$ contains no balanced strings.

For example if $T = \{3, 2, 2, 1, 1, 1, 0, 0\}$ then $T_1 = \{3, 2, 1, 0, -1, -2\}, T_2 = \{2, 1, 0, -1, -2\}$ and $T_3 = \{0\}$. If we require the strings to be balanced we have $T_0 = \{3, 2, 1, 0\}$ and $T_1 = \{2, 1, 0, -1, -2\}$.

We return to our set $S$, and first consider the set $S_0$. We write $S_0 = S_0' \cup S_{0,1} \cdots \cup S_{0,s}$ as a union of a set of maximal balanced strings as above, where $S_0'$ contains no balanced strings. Let $X = \{\#(S_{0,1}), \#(S_{0,1}), \ldots, \#(S_{0,s}), \#(S_{0,s})\}$ (each term counted twice).

Now write each set $S_0', S_1, \ldots, S_r$ as a disjoint union of strings. For each string $T$ which arises append $\#(T)$ to $X$.

Then $X$ is a set of positive integers. We write these in decreasing order and consider $X$ as a partition.
Now fix $G$ and let $\lambda = (a_1, \ldots, a_n)$ with $a_i \in \mathbb{R}$. If $G = SO(2n)$ with $n$ odd assume $a_i = 0$ for some $i$, i.e. $\lambda$ is $W$–conjugate to $-\lambda$. After conjugating by the Weyl group we may assume $a_1 \geq \ldots a_n \geq 0$. If $G = SO(2n)$ and $a_i \neq 0$ for all $i$ we may also need to apply an outer automorphism of $G$ to make $a_n > 0$; this is allowed since outer automorphisms preserve unitarity.

Let $S = \{a_1, \ldots, a_n\}$. Write $S = S_0 \cup S_1 \cup \cdots \cup S_r$ as above, where

$$S_0 = \begin{cases} \{a_i \in S \mid a_i \in \mathbb{Z}\} & G = Sp(n) \text{ or } SO(2n) \\ \{a_i \in S \mid a_i \in \mathbb{Z} + \frac{1}{2}\} & G = SO(2n+1) \end{cases}$$

Apply the above procedure to $S$. We obtain a partition $X$ which we denote $X(\lambda)$.

**Proposition 3.1** Fix $\lambda$.

1. The partition $X$ corresponds to a nilpotent orbit, denoted $O^\vee(\lambda)$, of $G^\vee$.

2. The map $\lambda \to O^\vee(\lambda)$ is a left inverse to the map $O^\vee \to \lambda(O^\vee) \colon O^\vee(\lambda(O^\vee)) = O^\vee$.

3. Let $h = \lambda(O^\vee(\lambda))$ and $M = M(O^\vee(\lambda))$. Then $\lambda = h + \nu$ where $\nu$ is the differential of the character of the center of $M$.

4. Let $H = H(O^\vee(\lambda))$. Then $X(\lambda)$ is unitary if and only if $\nu$ is in the $0$–complementary series of $H$. 