Signatures of Hermitian forms and unitary representations

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Outline

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Character formulas

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Introduction

$G(\mathbb{R})$ = real points of complex connected reductive alg G

Problem: find $\widehat{G(\mathbb{R})}_u =$ irr unitary reps of $G(\mathbb{R})$.

Harish-Chandra: $\widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})} =$ quasisimple irr reps.

Unitary reps $=$ quasisimple reps with pos def invt form.

Example: $G(\mathbb{R})$ compact $\Rightarrow \widehat{G(\mathbb{R})}_u = \widehat{G(\mathbb{R})} =$ discrete set.

Example: $G(\mathbb{R}) = \mathbb{R}$;

$$\widehat{G(\mathbb{R})} = \left\{\chi_z(t) = e^{zt} \ \ (z \in \mathbb{C})\right\} \simeq \mathbb{C}$$

$$\widehat{G(\mathbb{R})}_u = \{\chi_{i\xi} \ \ (\xi \in \mathbb{R})\} \simeq i\mathbb{R}$$

Suggests: $\widehat{G(\mathbb{R})}_u =$ real pts of cplx var $\widehat{G(\mathbb{R})}$. Almost...

$\widehat{G(\mathbb{R})}_h =$ reps with invt form: $\widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})}_h \subset \widehat{G(\mathbb{R})}$.

Approximately (Knapp): $\widehat{G(\mathbb{R})} =$ cplx alg var, real pts $\widehat{G(\mathbb{R})}_h$; subset $\widehat{G(\mathbb{R})}_u$ cut out by real algebraic ineqs.

Today: conjecture making inequalities computable.

Example: $SL(2,\mathbb{R})$ spherical reps

$G = SL(2,\mathbb{R}) = 2 \times 2$ real matrices of determinant 1

G acts on upper half plane $\mathbb{H} \rightsquigarrow$ repn $E(\nu)$ on $\nu^2 - 1$ eigenspace of Laplacian $\Delta_{\mathbb{H}}$.

Spectrum of $\Delta_{\mathbb{H}}$ on $L^2(\mathbb{H})$ is $(-\infty, -1] \leftrightsquigarrow \nu \in i\mathbb{R}$.

Most $E(\nu)$ irr; always unique irr subrep $J(\nu) \subset E(\nu)$.

Ex: $E(1) =$ harmonic fns on $\mathbb{H} \supset J(1) =$ constant fns

$J(\nu) \simeq J(\nu') \Leftrightarrow \nu = \pm\nu' \Rightarrow \widehat{G}_{sph} = \{J(\nu)\} \simeq \mathbb{C}/\pm 1$.

Cplx conj for real form of $\widehat{G}_{sph}$ is $\nu \mapsto -\overline{\nu}$; real points

$$\widehat{G}_{sph,h} \simeq (i\mathbb{R} \cup \mathbb{R})/\pm 1 \subset \mathbb{C}/\pm 1$$

These are sph Herm reps. Unitary pts (Bargmann):

$$\widehat{G}_{sph,u} \simeq (i\mathbb{R} \cup [-1,1])/\pm 1 \subset \mathbb{C}/\pm 1$$

Moral: have nice families of reps like $E(\nu)$; interesting irreducibles are smaller…

Categories of representations

G cplx reductive alg $\supset G(\mathbb{R})$ real form $\supset K(\mathbb{R})$ max cpt.

Rep theory of $G(\mathbb{R})$ modeled on Verma modules. . .

$H \subset B \subset G$ maximal torus in Borel subgp,

$\mathfrak{h}^* \leftrightarrow$ highest weight reps

$M(\lambda)$ Verma of hwt $\lambda \in \mathfrak{h}^*$, $L(\lambda)$ irr quot

Put cplxification of $K(\mathbb{R}) = K \subset G$, reductive algebraic.

$(\mathfrak{g}, K)$-mod: cplx rep V of $\mathfrak{g}$, compatible alg rep of K.

Harish-Chandra: irr $(\mathfrak{g}, K)$-mod $\leftrightsquigarrow$ "arb rep of $G(\mathbb{R})$."

X parameter set for irr $(\mathfrak{g}, K)$-mods

$I(x)$ std $(\mathfrak{g}, K)$-mod $\leftrightarrow x \in X$ $J(x)$ irr quot

Set X described by Langlands, Knapp-Zuckerman: countable union (subspace of $\mathfrak{h}^*$)/(subgroup of W).

Character formulas

Can decompose Verma module into irreducibles

$$M(\lambda) = \sum_{\mu \leq \lambda} m_{\mu,\lambda} L(\mu) \qquad (m_{\mu,\lambda} \in \mathbb{N})$$

or write a formal character for an irreducible

$$L(\lambda) = \sum_{\mu \leq \lambda} M_{\mu,\lambda} M(\mu) \qquad (M_{\mu,\lambda} \in \mathbb{Z})$$

Can decompose standard HC module into irreducibles

$$I(x) = \sum_{y \leq x} m_{y,x} J(y) \qquad (m_{y,x} \in \mathbb{N})$$

or write a formal character for an irreducible

$$J(x) = \sum_{y \leq x} M_{y,x} I(y) \qquad (M_{y,x} \in \mathbb{Z})$$

Matrices m and M upper triang, ones on diag, mutual inverses. Entries are KL polynomials eval at 1.

Forms and dual spaces

V cplx vec space (or alg rep of K, or $(\mathfrak{g}, K)$-mod).

Hermitian dual of V

$$V^h = \{\xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = \overline{z}\xi(v)\}$$

(If V is K-rep, also require ξ is K-finite.)

Sesquilinear pairings between V and W

$$\text{Sesq}(V, W) = \{\langle , \rangle \colon V \times W \to \mathbb{C}, \text{lin in } V, \text{conj-lin in } W\}$$

$$\text{Sesq}(V, W) \simeq \text{Hom}(V, W^h), \quad \langle v, w\rangle_T = (Tv)(w).$$

Cplx conj of forms is (conj linear) isom

$$\text{Sesq}(V, W) \simeq \text{Sesq}(W, V).$$

Corr (conj linear) isom is Hermitian transpose

$$\text{Hom}(V, W^h) \simeq \text{Hom}(W, V^h), \quad (T^h w)(v) = (Tv)(w).$$

Sesq form $\langle , \rangle_T$ Hermitian if

$$\langle v, v'\rangle_T = \overline{\langle v', v\rangle}_T \Leftrightarrow T^h = T.$$

Defining a rep on V^h

Suppose V is a $(\mathfrak{g}, K)$-module. Write π for repn map.

Want to construct functor

$$\text{cplx linear rep } (\pi, V) \rightsquigarrow \text{cplx linear rep } (\pi^h, V^h)$$

using Hermitian transpose map of operators. REQUIRES twisting by conjugate linear automorphism of $\mathfrak{g}$.

Assume

$$\sigma \colon G \to G \quad \text{antiholom aut}, \quad \sigma(K) = K.$$

Define $(\mathfrak{g}, K)$-module $\pi^{h,\sigma}$ on V^h,

$$\pi^{h,\sigma}(X) \cdot \xi = [\pi(-\sigma(X))]^h \cdot \xi \qquad (X \in \mathfrak{g}, \xi \in V^h).$$

$$\pi^{h,\sigma}(k) \cdot \xi = [\pi(\sigma(k)^{-1})]^h \cdot \xi \qquad (k \in K, \xi \in V^h).$$

Traditionally use

$$\sigma_0 = \text{ real form with complexified maximal compact } K.$$

We need also

$$\sigma_c = \text{ compact real form of } G \text{ preserving } K.$$

Invariant Hermitian forms

$V = (\mathfrak{g}, K)$-module, σ antihol aut of G preserving K.

A σ-invt sesq form on V is sesq pairing $\langle , \rangle$ such that

$$\langle X \cdot v, w \rangle = \langle v, -\sigma(X) \cdot w \rangle, \quad \langle k \cdot v, w \rangle = \langle v, -\sigma(k^{-1}) \cdot w \rangle$$

$$(X \in \mathfrak{g}; k \in K; v, w \in V).$$

Proposition

σ-invt sesq form on V $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-map $T \colon V \to V^{h,\sigma}$:

$$\langle v, w \rangle_T = (Tv)(w).$$

Form is Hermitian iff $T^h = T$.
Assume V is irreducible.
$V \simeq V^{h,\sigma} \Leftrightarrow \exists$ invt sesq form $\Leftrightarrow \exists$ invt Herm form
A σ-invt Herm form on V is unique up to real scalar.

$T \to T^h \leftrightsquigarrow$ real form of cplx line $\mathrm{Hom}_{\mathfrak{g},K}(V, V^{h,\sigma})$.

Invariant forms on standard reps

Recall multiplicity formula

$$I(x) = \sum_{y \leq x} m_{y,x} J(y) \qquad (m_{y,x} \in \mathbb{N})$$

for standard $(\mathfrak{g}, K)$-mod $I(x)$.

Want parallel formulas for σ-invt Hermitian forms.
Need forms on standard modules.

Form on irr $J(x) \xrightarrow{\text{deformation}}$ Jantzen filt $I_n(x)$ on std,
nondeg forms $\langle , \rangle_n$ on I_n/I_{n+1}.

Details (proved by Beilinson-Bernstein):

$$I(x) = I_0 \supset I_1 \supset I_2 \supset \cdots, \qquad I_0/I_1 = J(x)$$
$$I_n/I_{n+1} \text{ completely reducible}$$

$$[J(y) \colon I_n/I_{n+1}] = \text{coeff of } q^{(\ell(x)-\ell(y)-n)/2} \text{ in KL poly } Q_{y,x}$$

Hence $\langle , \rangle_{I(x)} \overset{\text{def}}{=} \sum_n \langle , \rangle_n$, nondeg form on gr $I(x)$.
Restricts to original form on irr $J(x)$.

Virtual Hermitian forms

$$\mathbb{Z} = \text{Groth group of vec spaces.}$$

These are mults of irr reps in virtual reps.

$$\mathbb{Z}[X] = \text{Groth grp of finite length reps.}$$

For invariant forms...

$$\mathbb{W} = \mathbb{Z} \oplus \mathbb{Z} = \text{Groth grp of fin diml forms.}$$

Ring structure

$$(p, q)(p', q') = (pp' + qq', pq' + q'p).$$

Mult of irr-with-forms in virtual-with-forms is in $\mathbb{W}$:

$$\mathbb{W}[X] \approx \text{Groth grp of fin lgth reps with invt forms.}$$

Two problems: invt form $\langle , \rangle_J$ may not exist for irr J; and $\langle , \rangle_J$ may not be preferable to $-\langle , \rangle_J$.

Hermitian KL polynomials: multiplicities

Fix σ-invt Hermitian form $\langle,\rangle_{J(x)}$ on each irr admitting one; recall Jantzen form $\langle,\rangle_n$ on $I(x)_n/I(x)_{n+1}$.

MODULO problem of irrs with no invt form, write

$$(I_n/I_{n-1}, \langle,\rangle_n) = \sum_{y \leq x} w_{y,x}(n)(J(y), \langle,\rangle_{J(y)}),$$

coeffs $w(n) = (p(n), q(n)) \in \mathbb{W}$; summand means

$$p(n)(J(y), \langle,\rangle_{J(y)}) \oplus q(n)(J(y), -\langle,\rangle_{J(y)})$$

Define Hermitian KL polynomials

$$Q^\sigma_{y,x} = \sum_n w_{y,x}(n) q^{(l(x)-l(y)-n)/2} \in \mathbb{W}[q]$$

Eval in $\mathbb{W}$ at $q = 1 \leftrightarrow$ form $\langle,\rangle_{I(x)}$ on std.

Reduction to $\mathbb{Z}[q]$ by $\mathbb{W} \to \mathbb{Z} \leftrightarrow$ KL poly $Q_{y,x}$.

Hermitian KL polynomials: characters

Matrix $Q^\sigma_{y,x}$ is upper tri, 1s on diag: INVERTIBLE.

$$P^\sigma_{x,y} \overset{\text{def}}{=} (-1)^{l(x)-l(y)}((x,y) \text{ entry of inverse}) \in \mathbb{W}[q].$$

Definition of $Q^\sigma_{x,y}$ says

$$(\text{gr } I(x), \langle , \rangle_{I(x)}) = \sum_{y \le x} Q^\sigma_{x,y}(1)(J(y), \langle , \rangle_{J(y)});$$

inverting this gives

$$(J(x), \langle , \rangle_{J(x)}) = \sum_{y \le x} (-1)^{l(x)-l(y)} P^\sigma_{x,y}(1)(\text{gr } I(y), \langle , \rangle_{I(y)})$$

Next question: how do you compute $P^\sigma_{x,y}$?

Herm KL polys for σ_c

$\sigma_c =$ cplx conj for cpt form of G, $\sigma_c(K) = K$.

Plan: study σ_c-invt forms, relate to σ_0-invt forms.

Proposition

Suppose $J(x)$ irr $(\mathfrak{g}, K)$-module, real infl char. Then $J(x)$ has σ_c-invt Herm form $\langle,\rangle^c_{J(x)}$, characterized by

$\langle,\rangle^c_{J(x)}$ is pos def on the lowest K-types of $J(x)$.

Proposition $\Longrightarrow$ Herm KL polys $Q^{\sigma_c}_{x,y}$, $P^{\sigma_c}_{x,y}$ well-def.

Coeffs in $\mathbb{W} = \mathbb{Z} \oplus s\mathbb{Z}$; $s = (0,1) \leftrightsquigarrow$ one-diml neg def form.

Conj: $Q^{\sigma_c}_{x,y}(q) = s^{\frac{\ell_o(x)-\ell_o(y)}{2}} Q_{x,y}(qs)$, $\quad P^{\sigma_c}_{x,y}(q) = s^{\frac{\ell_o(x)-\ell_o(y)}{2}} P_{x,y}(qs)$.

Equiv: if $J(y)$ appears at level n of Jantzen filt of $I(x)$, then Jantzen form is $(-1)^{(l(x)-l(y)-n)/2}$ times $\langle,\rangle_{J(y)}$.

Conjecture is false... but not seriously so. Need an extra power of s (shown in red) on the right side.

Orientation number

Conjecture $\leftrightarrow$ KL polys $\leftrightarrow$ *integral* roots.

Simple form of Conjecture $\Rightarrow$ Jantzen-Zuckerman translation across non-integral root walls preserves signatures of (σ_c-invariant) Hermitian forms.

It ain't necessarily so.

$SL(2,\mathbb{R})$: translating spherical principal series from (real non-integral positive) ν to (negative) $\nu - 2m$ changes sign of form iff $\nu \in (0,1) + 2\mathbb{Z}$.

Orientation number $\ell_o(x)$ is

1. # pairs $(\alpha, -\theta(\alpha))$ cplx nonint, pos on x; PLUS
2. # real β s.t. $\langle x, \beta^\vee \rangle \in (0,1) + \epsilon(\beta, x) + 2\mathbb{N}$.

$\epsilon(\beta, x) = 0$ spherical, 1 non-spherical.

Deforming to $\nu = 0$

Have computable conjectural formula (omitting ℓ_o)

$$(J(x), \langle , \rangle^c_{J(x)}) = \sum_{y \le x} (-1)^{l(x)-l(y)} P_{x,y}(s)(\mathrm{gr}\, I(y), \langle , \rangle^c_{I(y)})$$

for σ^c-invt forms in terms of forms on stds, same inf char.

Polys $P_{x,y}$ are KL polys: computed by `atlas`.

Std rep $I = I(\nu)$ deps on cont param ν. Put $I(t) = I(t\nu),\ t \ge 0$.

If std rep $I = I(\nu)$ has σ-invt form so does $I(t)$ ($t \ge 0$).

(signature for $I(t)$) $=$ (signature on $I(t + \epsilon)$), $\epsilon \ge 0$ suff small.

Sig on $I(t)$ differs from $I(t - \epsilon)$ on odd levels of Jantzen filt:

$$\langle , \rangle_{\mathrm{gr}\, I(t-\epsilon)} = \langle , \rangle_{\mathrm{gr}\, I(t)} + (s - 1) \sum_m \langle , \rangle_{I(t)_{2m+1}/I(t)_{2m+2}}.$$

Each summand after first on right is known comb of stds, all with cont param strictly smaller than $t\nu$. ITERATE...

$$\langle , \rangle^c_J = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle , \rangle^c_{I'(0)} \qquad (v_{J,I'} \in \mathbb{W}).$$

From σ_c to σ_0

Cplx conjs σ_c (compact form) and σ_0 (our real form) differ by Cartan involution θ: $\sigma_0 = \theta \circ \sigma_c$.

Irr $(\mathfrak{g}, K)$-mod $J \rightsquigarrow J^\theta$ (same space, rep twisted by θ).

Proposition

J admits σ_0-invt Herm form if and only if $J^\theta \simeq J$. If $T_0 \colon J \xrightarrow{\sim} J^\theta$, and $T_0^2 = \mathrm{Id}$, then

$$\langle v, w \rangle_J^0 = \langle v, T_0 w \rangle_J^c.$$

$T \colon J \xrightarrow{\sim} J^\theta \Rightarrow T^2 = z \in \mathbb{C} \Rightarrow T_0 = z^{-1/2} T \rightsquigarrow \sigma$-invt Herm form.

To convert formulas for σ_c invt forms $\rightsquigarrow$ formulas for σ_0-invt forms need intertwining ops $T_J \colon J \xrightarrow{\sim} J^\theta$, consistent with decomp of std reps.

Equal rank case

rk K = rk $G \Rightarrow$ Cartan inv inner: $\exists \tau \in K$, $\mathrm{Ad}(\tau) = \theta$.

$\theta^2 = 1 \Rightarrow \tau^2 = \zeta \in Z(G) \cap K$.

Study reps π with $\pi(\zeta) = z$. Fix square root $z^{1/2}$.

If ζ acts by z on V, and $\langle , \rangle_V^c$ is σ_c-invt form, then
$\langle v, w \rangle_V^0 \stackrel{\text{def}}{=} \langle v, z^{-1/2}\tau \cdot w \rangle_V^c$ is σ_0-invt form.

$$\langle , \rangle_J^c = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle , \rangle_{I'(0)}^c \qquad (v_{J,I'} \in \mathbb{W}).$$

translates to

$$\langle , \rangle_J^0 = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle , \rangle_{I'(0)}^0 \qquad (v_{J,I'} \in \mathbb{W}).$$

I' has LKT $\mu' \Rightarrow \langle , \rangle_{I'(0)}^0$ definite, sign $z^{-1/2}\mu(I')(t)$.

J unitary $\Leftrightarrow$ each summand on right pos def.

Computability of $v_{J,I'}$ needs conjecture about $P_{x,y}^{\sigma_c}$.

General case

Fix "distinguished involution" δ_0 of G inner to θ

Define extended group $G^\Gamma = G \rtimes \{1, \delta_0\}$.

Can arrange $\theta = \mathrm{Ad}(\tau\delta_0)$, some $\tau \in K$.

Define $K^\Gamma = \mathrm{Cent}_{G^\Gamma}(\tau\delta_0) = K \rtimes \{1, \delta_0\}$.

Study $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-mods V with $D_0 \colon V \xrightarrow{\sim} V^{\delta_0}$, $D_0^2 = \mathrm{Id}$.

Beilinson-Bernstein localization: $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ action of δ_0 on K-eqvt perverse sheaves on G/B.

Should be computable by mild extension of Kazhdan-Lusztig ideas. Not done yet!

Now translate σ_c-invt forms to σ_0 invt forms

$$\langle v, w\rangle_V^0 \stackrel{\text{def}}{=} \langle v, z^{-1/2}\tau\delta_0 \cdot w\rangle_V^c$$

on $(\mathfrak{g}, K^\Gamma)$-mods as in equal rank case.

Possible unitarity algorithm

Hope to get from these ideas a computer program; enter

- real reductive Lie group $G(\mathbb{R})$
- general representation π

and ask whether π is unitary.

Program would say either

- π has no invariant Hermitian form, or
- π has invt Herm form, indef on reps μ_1, μ_2 of K, or
- π is unitary, or
- I'm sorry Dave, I'm afraid I can't do that.

Answers to finitely many such questions $\rightsquigarrow$ complete description of unitary dual of $G(\mathbb{R})$.

This would be a good thing.