Galois and $\theta$ Cohomology

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Notes: www.liegroups.org/talks
Galois Cohomology

\(F \text{ local} \quad \Gamma = \text{Gal}(\overline{F}/F) \quad G = G(\overline{F}) \text{ defined over } F\)

\(G(F) = G(\overline{F})^{\Gamma}\)

\(H^i(\Gamma, G) = \text{Galois cohomology (group cohomology)}\)

\(i = 0, 1 \text{ if } G \text{ is not abelian}\)

Example: \(G(F) = \text{GL}(n, F) : \quad H^1(\Gamma, G) = 1\)

\((\text{GL}(1, F) = F^* : \text{Hilbert’s Theorem 90})\)

Example: \(G = \text{SO}(V) : \quad H^1(\Gamma, G) = \{ \text{non-degenerate quadratic forms of same dimension and discriminant as } V \}\)

Example: \(G = \text{Sp}(2n, F) \quad H^1(\Gamma, G) = \{ \text{non-degenerate symplectic forms, dim. } 2n \} = 1\)
Rational Forms

Basic Fact:

\[ \{ \text{rational (inner) forms of } G \} \leftrightarrow H^1(\Gamma, \text{G}_{\text{ad}}) \]

(Inner: \( \sigma'\sigma^{-1} \) is inner)

NB: (for the experts): equality of rational forms is by the action of \( G \), not \( \text{Aut}(G) \) (Borel: \( \text{Image}(H^1(\Gamma, \text{G}_{\text{ad}})) \rightarrow H^1(\Gamma, \text{Aut}(\text{G}_{\text{ad}})) \))

Theorem (Kneser): \( F \) p-adic, \( G \) simply connected \( \Rightarrow H^1(\Gamma, G) = 1 \)
Not true over \( \mathbb{R} \) ... \( G(\mathbb{R}) = SU(2), \quad H^1(\Gamma, G) = \mathbb{Z}/2\mathbb{Z} \)

Problem: Calculate \( H^1(\Gamma, G) \) \( G \) simply connected, defined over \( \mathbb{R} \)
This fact plays a role in statements about the trace formula, functoriality, packets...
Real case

\[ F = \mathbb{R}, \Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle \]

\[ H^0(\Gamma, G) = G(\mathbb{R}) \]

\[ H^1(\Gamma, G) = \{ g \in G \mid g\sigma(g) = 1 \}/[g \to xg\sigma(x^{-1})] \]

Write \( H^i_\sigma(\Gamma, G) \)

**Digression:** \( H = \text{torus}, \quad \widehat{H}^i(\Gamma, H) \) Tate cohomology

\[ \widehat{H}^0(\Gamma, H) = H(\mathbb{R})/H(\mathbb{R})^0 \]

Question: Is it possible (and a good idea?) to define \( \widehat{H}^i(\Gamma, G) \) \((i = 0, 1)\) in such a way that \( \widehat{H}^0(\Gamma, G) = G(\mathbb{R})/G(\mathbb{R})^0? \)
Real Forms: $\sigma$ and $\theta$ pictures

Cartan: classify real forms by their **Cartan involution**: a real form is determined by its maximal compact subgroup $K(\mathbb{R})$ - in fact by $K(\mathbb{C}) = G(\mathbb{C})^\theta$

$\theta$ is a **holomorphic** involution.

\[
\begin{array}{ccc}
K(\mathbb{C}) = G(\mathbb{C})^\theta & \xrightarrow{\theta} & G(\mathbb{C}) \\
& & \xrightarrow{\sigma} \\
G(\mathbb{R}) = G(\mathbb{C})^\sigma & \xleftarrow{\theta} & K(\mathbb{R}) = G(\mathbb{R})^\theta = K(\mathbb{C})^\sigma
\end{array}
\]
Real Forms: $\sigma$ and $\theta$ pictures

\[ \begin{array}{ccc}
GL(n, \mathbb{C}) & \sigma & GL(n, \mathbb{C}) \\
\theta & & \theta
\end{array} \]

\[ GL(p, \mathbb{C}) \times GL(q, \mathbb{C}) \]

\[ \begin{array}{ccc}
GL(p, \mathbb{C}) \times GL(q, \mathbb{C}) & U(p, q) & U(p) \times U(q) \\
\sigma & \sigma & \theta
\end{array} \]

Theorem: (Cartan)

\[ \{ \sigma \mid \sigma \text{ antiholomorphic} \} / G \leftrightarrow \{ \theta \mid \sigma \text{ holomorphic} \} / G \]

\[ \sigma \rightarrow \theta = \sigma \sigma_c \]

\[ \sigma = \theta \sigma_c \leftarrow \theta \]
Real Forms: $\sigma$ and $\theta$ pictures

$\sigma$, $\theta$ pictures are deeply embedded in representation theory

$\sigma : G(\mathbb{R})$ acting on a Hilbert space

$\theta : (\mathfrak{g}, K)$ modules $\mathfrak{g}$, $K$ both complex

Matsuki duality (later): $X = G(\mathbb{C})/B(\mathbb{C})$

$$G(\mathbb{R}) \backslash X \leftrightarrow K(\mathbb{C}) \backslash X$$

Kostant-Sekiguchi correspondence (nilpotent $G(\mathbb{R})$, $K(\mathbb{C})$ orbits)
Real Forms: $\theta$ cohomology

$\theta$ holomorphic:

**Definition**: $H^i_\theta(\mathbb{Z}_2, G)$: group cohomology with $\mathbb{Z}_2$ acting by $\theta$

$$H^0_\theta(\mathbb{Z}_2, G) = K$$

(remember $K = K(\mathbb{C})$)

$$H^1_\theta(\mathbb{Z}_2, G) = \{g \mid g\theta(g) = 1\}/[g \to xg\theta(x^{-1})]$$

Basic Point: $H^1_\theta(\mathbb{Z}_2, G)$ is much easier to compute than $H^1_\sigma(\Gamma, G)$

**Example**: $\theta = 1$:

$$H^1_\theta(\mathbb{Z}_2, G) = \{g \mid g^2 = 1\}/G = \{h \in H \mid h^2 = 1\}/W$$

$G(\mathbb{R})$ compact
(Serre): $H^1(\Gamma, G) = H^1(\Gamma, G(\mathbb{R})) = \{h \in H(\mathbb{R}) \mid h^2 = 1\}/W$
Real Forms: Example

\( \theta = 1, \ H^1(\mathbb{Z}_2, G) = H_2/W : \)

Exercise:

\( G = E_8, \ R=\text{root lattice}, \ |R/2R| = 256 \)

\[ |H^1_{\theta}(\mathbb{Z}_2, G)| = |(R/2R)/W| = 3 \quad (1 + 120 + 135 = 256) \]
Galois and $\theta$ cohomology

$$H^1_\theta(\mathbb{Z}_2, G) \quad H^1_\sigma(\Gamma, G)$$

Cartan’s Theorem can be stated: $\sigma \leftrightarrow \theta \Rightarrow$

$$H^1_\sigma(\Gamma, G_{ad}) \cong H^1_\theta(\mathbb{Z}_2, G_{ad})$$

**Question:** drop the adjoint condition?

**Theorem:** $G$ connected reductive,

$\sigma$ antiholomorphic, $\theta$ holomorphic

$\sigma \leftrightarrow \theta$ (in the sense of Cartan; i.e. defining the same real form)

There is a canonical isomorphism:

$$H^1_\sigma(\Gamma, G) \cong H^1_\theta(\mathbb{Z}_2, G)$$
Sketch of proof

(1) $H$ torus: $1 \to H_2 \to H \xrightarrow{\mathbb{Z}_2} H \to 1$

$|\Gamma| = 2 \Rightarrow$

$H^1_{\sigma}(\Gamma, H) \simeq H^1_{\sigma}(\Gamma, H_2)$

$H^1_{\theta}(\mathbb{Z}_2, H) \simeq H^1_{\theta}(\mathbb{Z}_2, H_2)$

and $\theta|_{H_2} = \sigma|_{H_2}$

$$H^1_{\sigma}(\Gamma, H) \simeq H^1_{\sigma}(\Gamma, H_2) = H^1_{\theta}(\mathbb{Z}_2, H_2) \simeq H^1_{\theta}(\mathbb{Z}_2, H)$$

(2) $H_f$ a fundamental (most compact) Cartan subgroup;

$$H^1_{\sigma}(\Gamma, H_f) \to H^1_{\sigma}(\Gamma, G)$$

(easy: every semisimple elliptic element is conjugate to an element of $H_f$)
Sketch of proof (continued)

(3) $W_i(H) = $ Weyl group of imaginary roots,

$$H^1_{\sigma}(\sigma, H) / W_i(H) \hookrightarrow H^1_{\sigma}(\Gamma, G)$$

This is non-trivial but standard:

it comes down to $(G/P)(F) = G(F)/P(F))$ (Borel-Tits) and there is only one conjugacy class of compact Cartan subgroups (very special to $\mathbb{R}$)

Equivalently: over $\mathbb{R}$ stable conjugacy of Cartan subgroups is equivalent to ordinary conjugacy (Shelstad) (false in the $p$-adic case)

$$H^1_{\sigma}(\Gamma, G) \simeq H^1_{\sigma}(\Gamma, H_f) / W_i$$

Theorem (Borovoi): $H^1_{\sigma}(\Gamma, G) \simeq H^1_{\sigma}(\Gamma, H_f) / W_{\sigma}$

Exactly same argument holds for $\theta$-cohomology:

$$H^1_{\theta}(\mathbb{Z}_2, G) \simeq H^1_{\theta}(\mathbb{Z}_2, H_f) / W_i$$
Applications

Two versions of the rational Weyl group

\[ W_\sigma = \text{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R}) \]

\[ W_\theta = \text{Norm}_{K(\mathbb{C})}(H(\mathbb{C}))/H(\mathbb{C}) \cap K(\mathbb{C}) \]

Theorem (well known, see Warner): \( W_\sigma \simeq W_\theta \)
Applications

proof:

\[ 1 \to H \to N \to W \to 1 \]

\[ 1 \to H^\sigma \to N^\sigma \to W^\sigma \to H^1_{\sigma}(\Gamma, H) \]

\[ 1 \longrightarrow W_\sigma = N^\sigma / H^\sigma \longrightarrow W^\sigma \longrightarrow H^1_{\sigma}(\Gamma, H) \]

\[ 1 \longrightarrow W_\theta = N^\theta / H^\theta \longrightarrow W^\theta \longrightarrow H^1_{\theta}(\Gamma, H) \]
Applications

Matsuki Correspondence of Cartan subgroups

Theorem (Matsuki): There is a canonical bijection

\[ \{ \sigma\text{-stable } H \}/G(\mathbb{R}) \leftrightarrow \{ \theta\text{-stable } H \}/K \]

Proof in quasisplit case:

\[ \text{LHS} = H^1_\sigma(\Gamma, W) \cong H^1_\theta(\mathbb{Z}_2, W) = \text{RHS} \]

(general: \( H^1_\sigma(\Gamma, N) \cong H^1_\theta(\mathbb{Z}_2, N) \ldots \))
Applications: Strong real forms

For simplicity: assume equal rank inner class

Definition (ABV) A strong real form of $G$ is $G$-conjugacy class of $x \in G$ satisfying $x^2 \in Z(G)$.

$\{\text{strong real forms}\} \rightarrow \{\text{real forms}\}$ (bijection if $G$ is adjoint)

$x \rightarrow \theta_x = \text{int}(x)$ (conjugation by $x$)

Pure Real forms: $x^2 = 1$

Problem:

1) Give a cohomological definition of strong real forms
2) Define “strong rational forms” of p-adic groups (Kaletha): $H^1(u \rightarrow W, Z \rightarrow Z) =$strong real forms in real case
Applications: Strong real forms

Strong Real Forms:

\[ x \rightarrow \text{inv}(x) = x^2 \in \mathbb{Z}^\Gamma \]

Real forms:

\[ \text{inv} : H^1(\Gamma, G_{ad}) \rightarrow H^2(\Gamma, \mathbb{Z}) = \hat{H}^0(\Gamma, \mathbb{Z}) = \mathbb{Z}^\Gamma/(1 + \sigma)\mathbb{Z} \]

**Theorem:** Given \( \sigma \rightarrow \text{inv}(\sigma) \in \mathbb{Z}^\Gamma/(1 + \sigma)\mathbb{Z} \rightarrow (\text{choose}) \ z \in \mathbb{Z}^\Gamma \)

\[ H^1(\Gamma, G) \iff \{\text{strong real forms of type } z\} \]

\( \rightarrow \): classical Galois cohomology interpretation of strong real forms

\( \leftarrow \): compute \( H^1(\Gamma, G) \)(the right hand side is easy)
Applications: Strong real forms

Corollary:

\[ \{\text{strong real forms}\} \leftrightarrow \bigcup_{z \in S} H^1_{\sigma_z}(\Gamma, G) \]

\[ S = \mathbb{Z}/(1 + \sigma)\mathbb{Z} \]

\[ S \ni z \rightarrow \sigma_z (\sigma_z \leftrightarrow \theta_x \rightarrow x^2 = z) \]
Application: Computing $H^1(\Gamma, G)$

Compute $\{\text{strong real forms of type } z\}$

(equal rank case):

$$H^1(\Gamma, G) \simeq \{g \in G \mid g^2 = z\}/G = \{h \in H \mid h^2 = z\}/W$$

($z$ depends on the real form)

Example:

$G = Sp(2n, \mathbb{R})$ $x = \text{diag}(i, \ldots, i, -i, \ldots, -i)$ $z = -I$

$$H^1_{\sigma}(\Gamma, G) = \{g \mid g^2 = -I\}/G = \{\text{diag}(\pm i, \ldots, \pm i)\}/W = 1$$

Example:

$G = Spin(p, q)$

$SO(p, q): \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor + 1$ (classifying quadratic forms)

$Spin(p, q): \left\lfloor \frac{p+q}{4} \right\rfloor + \delta(p, q)$ $\delta(p, q) = 0, 1, 2, 3$