Atlas of Lie Groups
and Representations

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Calculating the Hodge Filtration
or
Hermitian Forms and Hodge Theory

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The Main Result

Joint with Peter Trapa, David Vogan

$G(\mathbb{R})$: a real form of a connected, complex reductive group

π: irreducible representation

Main Theorem

The signature of the c-form on π is
the reduction mod(2) of the Hodge filtration

Today:

(1) What does this mean?
(2) What does this *mean*?
(3) Relationship with the Schmid-Vilonen conjecture

Hermitian forms $\longleftrightarrow$ Hodge theory

K-multiplicities

$G(\mathbb{C}), G(\mathbb{R}), \theta,\ K = G^\theta,\ \mathfrak{g} = \mathrm{Lie}(G)$

π admissible $(\mathfrak{g}, K)$-module

$$\pi|_K = \sum_{\mu \in \widehat{K}} \mathrm{mult}_\pi(\mu)\mu$$

Theorem: There is an algorithm to compute $\mathrm{mult}_\pi(\mu)$

Morally this comes down to the Blattner formula plus parabolic induction. Practically speaking is an entirely different matter (for one thing K is disconnected). This algorithm has been implemented in the `Atlas` software.

A few words about $\widehat{K}$

From now on every representation has real infinitesimal character:

$$\lambda \in X^* \otimes \mathbb{R} \quad \text{(via the Harish-Chandra homomorpism)}$$

Suppose $P = MAN$ is a (real) parabolic subgroup, π_M is a discrete series of M, and $\nu \in \mathfrak{a}^*$.

$\mathrm{Ind}_P^G(\pi_M \otimes \nu)$:

has real infinitesimal character: $\nu \in \mathfrak{a}_0^*$ (real vector space)

is tempered: $\nu \in i\mathfrak{a}_0^*$

is tempered with real infinitesimal character: $\nu = 0$

(countable set)

A few more words about $\widehat{K}$

$\mathcal{P}_{\text{temp}}$: $\{\pi \mid$ irreducible, tempered (real inf. char.)$\}$

Theorem (Vogan):

Bijection:

$$\mathcal{P}_{\text{temp}} \longleftrightarrow \widehat{K}$$

$$\pi \to \text{lowest K-type of } \pi$$

Note: If X is a $(\mathfrak{g}, K)$-module of finite length, then

$$\text{mult}_X = \sum_{i=1}^{n} a_i \text{mult}_{\pi} \quad (a_i \in \mathbb{Z}, \pi_i \in \mathcal{P}_{\text{temp}})$$

Example

$G(\mathbb{R}) = SL(2,\mathbb{R})$

$K = S^1, \widehat{K} = \mathbb{Z}$

$\mathbb{C}$=trivial representation of $SL(2,\mathbb{R})$:

$$\text{(reducible) spherical principal series} = \mathbb{C} + \text{DS}_+ + \text{DS}_-$$

$$\mathbb{C}|_K = \text{spherical principal series}|_K - \text{DS}_+|_K - \text{DS}_-|_K$$

$$2\mathbb{Z} - \{2,4,6,\dots\} - \{-2,-4,-6,\dots\}$$

PS: spherical principal series with infinitesimal character 0

$\text{DS}_\pm$: holomorphic/antiholomorphic discrete series with infinitesimal character ρ

$$\text{mult}_{\mathbb{C}} = \text{mult}_{PS} - \text{mult}_{DS_+} - \text{mult}_{DS_+}$$

Signatures of Hermitian forms

$G, \theta, K \dots G(\mathbb{R}) = G^\sigma$ (σ antiholomorphic)

Suppose (π, V) admits an invariant Hermitian form:

$$\langle \pi(X)v, w\rangle + \langle v, \pi(\sigma(X))w\rangle = 0$$

Theorem: an irreducible representation π of $G(\mathbb{R})$ is unitary if and only if its $(\mathfrak{g}, K)$-module admits a positive definite invariant Hermitian form.

Problem: Describe the Unitary Dual

set of equivalence classes of irreducible unitary representations

Signatures of Hermitian Forms

Problem: Suppose (π, V) supports an invariant Hermitian form $\langle\, , \rangle$. Compute the signature of $\langle\, , \rangle$.

What? $\langle\, , \rangle$ is positive definite if $\langle v, v\rangle > 0$ for all v

If not, what is the “signature”?

Definition: $\mathbb{W} = \mathbb{Z}[z]/(z^2 - 1) = \mathbb{Z}[s]$ $(s^2 = 1)$

Definition: $\mathrm{sig}_\pi : \widehat{K} \to \mathbb{W}$:

$\mathrm{sig}_\pi(\mu) = a + bs$ if in the invariant form, restricted to the K-isotypic, μ occurs a (resp. b) times with positive (resp. negative) definite form.

Note: $\mathrm{sig}_\pi(\mu)(s = 1) = \mathrm{mult}_\pi(\mu)$

The question becomes: how to “compute” sig_π?

Signatures of Hermitian Forms

Theorem: $\text{sig}_\pi = \sum_{i=1}^{n} w_i \text{mult}_{\pi_i}$ for some irreducible, tempered representations $\pi_1, \ldots, \pi_n$, $w_i \in \mathbb{W}$

The point is this is a finite formula.

In other words

$$\text{sig}_\pi \in \mathbb{W}\langle \text{mult}_\tau \mid \tau \text{ tempered } \rangle$$

Example: $SL(2,\mathbb{R})$

$\pi(\nu)$: spherical principal series with infinitesimal character $\nu \in \mathbb{R}$

$\widehat{K} = \mathbb{Z}$

$\pi(\nu)|_K = 2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$

$\pi(\nu)$ is reducible $\Leftrightarrow \nu \in 2\mathbb{Z}+1$

$\mathrm{sig}_{\pi(0)} = \mathrm{mult}_{\pi(0)}$ (unitary)

in fact

$\mathrm{sig}_{\pi(\nu)} = \mathrm{sig}_{\pi(0)} = \mathrm{mult}_{\pi(0)} \quad \nu < 1$

	...	-6	-4	-2	0	2	4	6	...
sig(I(0))		+	+	+	+	+	+	+	
sig(I($1-\epsilon$))		+	+	+	+	+	+	+	
sig(I(1))		0	0	0	+	0	0	0	
sig(I($1+\epsilon$))		-	-	-	+	-	-	-	

Signatures of Hermitian forms

Conclusion:

$$\mathrm{sig}_{\pi(1+\epsilon)} = \mathrm{mult}_{\pi(1-\epsilon)} + (s-1)(\mathrm{mult}_{\pi}(DS_{+}) + \mathrm{mult}_{\pi}(DS_{-}))$$

$=$all positive signs...change signs from $+$ to -

The c-form

Major fly in the ointment:

a) there may be no invariant Hermitian form on (π, V)
b) it may not be unique (up to positive scalar)

Example: odd principal series of $SL(2,\mathbb{R})$ with $\nu \neq 0$

The K-types $1, -1$ have opposite signature

$G(\mathbb{R}), \sigma \quad \sigma_c$ compact real form (so $\sigma_c \circ \sigma = \theta$)

Definition The c-form satisfies

$$\langle \pi(X)v, w\rangle_c + \langle v, \pi(\sigma_c(X))w\rangle_c = 0$$

and $\langle\ ,\ \rangle_c$ is positive definite on all lowest K-types

The c-form

Theorem:
(1) The c-form exists and is unique (up to positive scalar)

(2) The c-form determines the invariant Hermitian form (an explicit formula)

Note: if the group is not equal rank we need the c-form on the *extended* group

Definition: $\mathrm{sig}^c_\pi : \widehat{K} \to \mathbb{W}$:

$\mathrm{sig}^c_\pi(\mu) = a + bs$ if in the c-form, restricted to the K-isotypic, μ occurs a (resp. b) times with positive (resp. negative) definite form.

Same result as before: $\mathrm{sig}^c_\pi = \sum_i w_i^c \mathrm{mult}_{\pi_i}$

Digression: The Langlands classification and the KLV polynomials

Fix infinitesimal character λ

$\mathcal{P}_\lambda$: a set of parameters

$\mathcal{P}_\lambda \ni \gamma \to I(\gamma)$ (standard module)

$\mathrm{J}(\gamma)$ (unique irreducible quotient of $I(\gamma)$)

$\{$irreducible representations with infinitesimal character $\gamma\} \longleftrightarrow \mathcal{P}_\lambda$

$$I(\gamma) = \mathrm{Ind}_{MAN}^{G}(\pi_M \otimes \nu \otimes 1) \quad (\nu \in \mathfrak{a}_0^*)$$

Deformation: $\gamma_t \leftrightarrow \mathrm{Ind}_{MAN}^{G}(\pi_M \otimes t\nu \otimes 1)$

Digression: The Langlands classification and the KLV polynomials

Kazhdan-Lusztig-Vogan polynomials:

$P_{\tau,\gamma} \in \mathbb{Z}[q]$

$\{P_{\tau,\gamma} \mid \tau, \gamma \in \mathcal{P}_\lambda\}$ (upper unitriangular matrix)

Inverse matrix $\{Q_{\tau,\gamma}\}$ (with signs)

$$\mathsf{J}(\gamma) = \sum_\tau (-1)^{\ell(\gamma)-\ell(\tau)} P_{\tau,\gamma}(1)\mathsf{I}(\tau)$$

$$\mathsf{I}(\gamma) = \sum_\tau Q_{\tau,\gamma}(1)\mathsf{J}(\tau)$$

Digression: the Jantzen filtration

$$\mathrm{I}(\gamma) = \sum_{\tau} Q_{\tau,\gamma} J(\tau)$$

The Jantzen filtration is a canonical filtration of $\mathrm{I}(\gamma)$ by $(\mathfrak{g}, K)$-modules.

Jantzen conjecture: if $Q_{\tau,\gamma} = \sum a_j q^j$, then
a_r is the multiplicity of $\mathrm{J}(\tau)$ in level $\frac{1}{2}(\ell(\gamma) - \ell(\tau) + r)$ of the Jantzen filtration.

Note: $Q_{\tau,\gamma}(1) = \sum_r a_r$ is the multiplicity of $\mathrm{J}(\tau)$ in $\mathrm{I}(\gamma)$.

Computing the c-form

Suppose $\mathsf{I}(\gamma)$ is a reducible standard module (at some ν), and $\mathsf{I}(\gamma_t)$ is irreducible for $0 < |1-t| < \epsilon$.

$$\mathsf{I}(\gamma_{1-\epsilon}) \to \mathsf{I}(\gamma_1) \to \mathsf{I}(\gamma_{1+\epsilon})$$

Problem: how does the c-form change as you deform from $\mathsf{I}(\gamma_{1-\epsilon})$ to $\mathsf{I}(\gamma_{1+\epsilon})$?

Key fact: the c-form changes sign on odd levels of the Jantzen filtration at $\mathsf{I}(\Gamma)$

(Comes down to: $f(x) = x^n$ changes sign at $x = 0$ if and only if n is odd.)

Computing the c-form

Algorithm (Deformation of the c-form):

$$\mathrm{sig}(\gamma_{1+\epsilon}) = \mathrm{sig}(\gamma_{1-\epsilon}) + (1-s) \sum_{\substack{\phi,\tau \\ \phi<\tau<\gamma \\ \ell(\gamma)-\ell(\tau)\ \text{odd}}} s^{(\ell_0(\gamma)-\ell_0(\tau))/2} P_{\phi,\tau}(s) Q_{\tau,\gamma}(s) \mathrm{sig}(\mathrm{I}(\phi))$$

Corollary

There is an inductive algorithm to compute sig($I(\gamma)$), in terms of sig($I(\phi)$) where $I(\phi)$ is (irreducible) tempered.

The Hodge filtration

Saito's theory of mixed Hodge modules.

Beilinson-Bernstein theory of $\mathcal{D}$-modules, $\mathcal{D}_\lambda$-modules

Global section functor: equivalence of categories $\mathcal{D}_\lambda$-modules and $(\mathfrak{g}, K)$-modules with infinitesimal character λ.

The Hodge filtration

Schmid/Vilonen:

Theorem If π is an irreducible or standard $(\mathfrak{g}, K)$-module (π, V) it has the following canonical constructions:

1) Finite, ascending weight filtration by $(\mathfrak{g}, K)$-modules (the Jantzen filtration) $W_0 \subset W_1 \cdots \subset W_n = V$

2) Infinite, ascending Hodge filtration by finite dimensional K-modules $F_0 \subset F_1 \subset F_2 \dots$

Caveat: Schmid and Vilonen have not published a proof of this (need: the global section functor is filtered exact)

The Hodge filtration

$(\pi, V) \quad 0 \subset F_0 \subset F_1 \subset \dots$

$\mathrm{gr}(\pi) = F_p/F_{p-1}$ (a finite dimensional representation of K)

Definition: $\mathrm{hodge}_\pi : \widehat{K} \to \mathbb{Z}[v]$

$\mathrm{hodge}_\pi(\mu) = a_0 + a_1 v + \dots + a_n v^n$: $\quad a_i = \mathrm{mult}_{\mathrm{gr}_i(\pi)}(\mu)$

The Hodge Filtration

So:

$$\begin{aligned} \mathrm{hodge}_\pi &: \widehat{K} \to \mathbb{Z}[v] \\ \mathrm{sig}^c_\pi &: \widehat{K} \to \mathbb{Z}[s] \\ \mathrm{mult}_\pi &: \widehat{K} \to \mathbb{Z} \end{aligned}$$

Note:

$$\mathrm{hodge}_\pi|_{v=1} = \mathrm{sig}^c_\pi|_{s=1} = \mathrm{mult}_\pi$$

Examples of the Hodge filtration

$SL(2,\mathbb{R})$, $\pi(0)$ = tempered, spherical principal series,
$V = \langle w_k \mid k \in 2\mathbb{Z}\rangle$.

$\text{hodge}_{I(0)}(w_{2k}) = v^{|k|}$

$G(\mathbb{R})$ split, $I(0)$: $I(0)|_K \simeq$ ring of regular functions on $\mathcal{N} \cap \mathfrak{p}$

Discrete series: graded Blattner formula

The Main Result

Theorem (Adams/Trapa/Vogan):

$$\mathrm{hodge}_\pi|_{v=s} = \mathrm{sig}^c_\pi$$

In other words: if $\mu \in \widehat{K}$:

$$\mathrm{hodge}_\pi(\mu) = a_0 + a_1 v + \cdots + a_n v^n$$

implies

$$\begin{aligned}\mathrm{sig}^c_\pi(\mu) &= a_0 + a_1 s + a_2 s^2 + \ldots a_n s^n \\ &= (a_0 + a_2 + \ldots) + (a_1 + a_3 + \ldots)s\end{aligned}$$

Sketch of proof

From earlier:

Suppose $\mathsf{I}(\gamma)$ is a reducible standard module (at some ν), and $\mathsf{I}(\gamma_t)$ is irreducible for $0 < |1 - t| < \epsilon$.

Problem: how does the c-form change as you deform from $\mathsf{I}(\gamma_{1-\epsilon})$ to $\mathsf{I}(\gamma_{1+\epsilon})$?

Key fact (signature): the c-form changes sign on odd levels of the Jantzen filtration.

Problem: how does the Hodge filtration change as you deform from $\mathsf{I}(\gamma_{1-\epsilon})$ to $\mathsf{I}(\gamma_{1+\epsilon})$?

Key fact (Hodge): a K-type in level k of the Jantzen filtration jumps by k levels in the Hodge filtration.

Sketch of Proof

Algorithm (Deformation of the c-form):

$$\mathrm{sig}(\gamma_{1+\epsilon}) = \mathrm{sig}(\gamma_{1-\epsilon}) + (1-s) \sum_{\substack{\phi,\tau \\ \phi<\tau<\gamma \\ \ell(\gamma)-\ell(\tau)\ \mathrm{odd}}} s^{(\ell_0(\gamma)-\ell_0(\tau))/2} P_{\phi,\tau}(s) Q_{\tau,\gamma}(s)\mathrm{sig}(I(\phi))$$

Algorithm (Deformation of the Hodge filtration):

$$\mathrm{hodge}(I(\Gamma_{1+\epsilon})) = \mathrm{hodge}(I(\Gamma_{1-\epsilon})) - \sum_{\Phi<\Gamma} v^{(\ell_0(\Gamma)-\ell_0(\Phi)/2} \left[\sum_{\substack{\Xi \\ \Phi\leq\Xi\leq\Gamma}} (-1)^{\ell(\Xi)-\ell(\Phi)} v^{\ell(\Gamma)-\ell(\Xi)} P_{\Phi,\Xi}(v) Q_{\Xi,\Gamma}(v^{-1}) \right] \mathrm{hodge}(I(\Phi))$$

Sketch of proof

The Hodge formula, evaluated at $v = s$, gives the signature formula.

This reduces us to the case of tempered representations.

[This is another story about as long as this one]

Caveat: We haven't completely finished the tempered part of the argument.

Note: This is *theorem*. It *also* gives an algorithm to compute the Hodge filtration.

The Schmid-Vilonen Conjecture

Conjecture 1: The c-form restriced to F_p is non-degenerate

Assuming this the c-form induces a form on

$$\mathrm{gr}_p(\pi) = F_p/F_{p-1} \simeq F_p \cap F_{p-1}^{\perp}$$

Conjecture 2: The c-form on $\mathrm{gr}_p(\pi)$ is definite of sign $\epsilon_\pi(-1)^p$

($\epsilon_\pi = \pm 1$ is an elementary sign)

Conjecture 2 implies the Main Theorem

(but NOT vice-versa)

(Main Theorem $+$ Conjecture 1 $\not\Rightarrow$ Conjecture 2😥)

Thank You