The Real Chevalley Involution

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The Chevalley Involution

G: connected, reductive, H : Cartan subgroup

Theorem

(1) There is an involution C of G satisfying: $C(h) = h^{-1}$ $(h \in H)$;

(2) $C(g) \sim g^{-1}$ for all semisimple elements g;

(3) Any two such involutions are conjugate by an inner automorphism;

C is the Chevalley involution of G

Example: $G = GL(n), SL(n)$:, $C(g) = {}^t g^{-1}$ (outer)

Example: C is inner $\Leftrightarrow -1 \in W$

C is the Cartan involution of the split real form of $G(\mathbb{C})$.

The Contragredient

(π, V)

$V^* = \mathrm{Hom}(V, \mathbb{C})$

$\pi^*(g)(f)(v) = f(g^{-1}v)$

Character: $\theta_{\pi^*}(g) = \theta_\pi(g^{-1})$

On the Dual Side

G defined over F (local)

$\phi : W'_F \to {}^LG \twoheadrightarrow \Pi(\phi)$ (L-packet)

What is the effect of $\phi \to C \circ \phi$?

$\pi^*, \Pi(\phi)^*$: contragredient

Theorem (A/Vogan)

$F = \mathbb{R}$:

$$\boxed{\Pi(C \circ \phi) = \Pi(\phi)^*}$$

(Mumbai 2012, arXiv 1201.0496)

(Conjecturally true over arbitrary F).

When is every L-packet self-dual?

Corollary

Every L-packet is self-dual if and only if $-1 \in W(G, H)$

$(W(G, H) = W(G(\mathbb{C}), H(\mathbb{C})))$

The Group side

What is the effect of the Chevalley automorphism on the group side?

Question

(1) Is C defined over F?

(2) Does it satisfy $\pi^C \simeq \pi^*$?

Character:

(2′) $C(g) \sim_{G(F)} g^{-1}$ for all $g \in G(F)$?

Note: (1) $\Rightarrow C(g) \sim_{G(\overline{F})} g^{-1}$

Motivation

General question: automorphisms of G, (e.g. outer involutions), effect on representations, also on the dual side

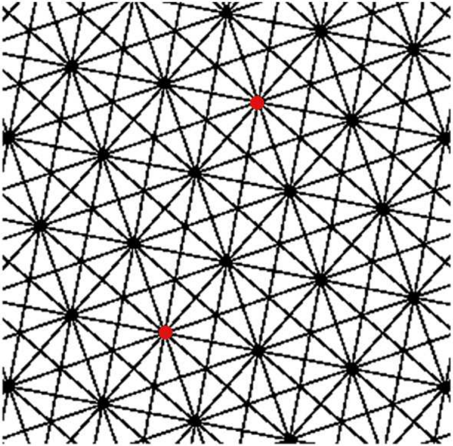
Character theory, relation with automorphisms

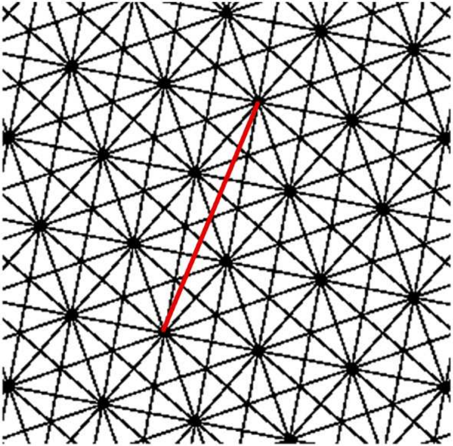
Frobenius-Schur (symplectic/orthogonal) indicator

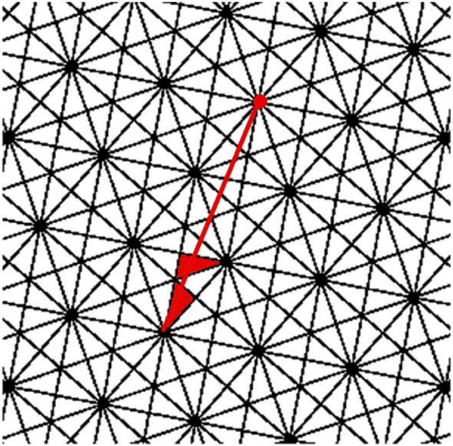
Applications to L-functions (contragredient)

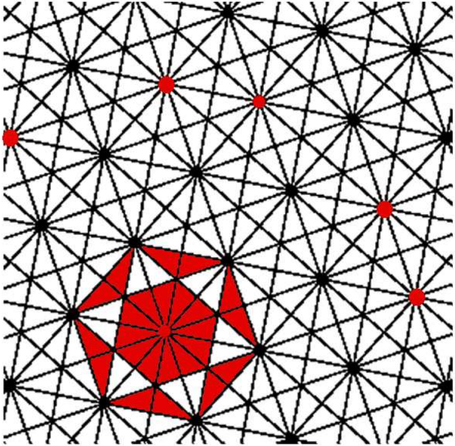
recent paper of D. Prasad and Ramakrishnan

Hermitian dual, (closely related to an automorphism on the space of representations), applications to unitarity









The Contragredient

Example (D. Prasad)

$G = F_4, G_2, E_8$, F p-adic, $G(F)$ split

There are Chevalley involutions C of G, defined over F

None of them satisfy: $C(g) \sim_{G(F)} g^{-1}$

(only $C(g) \sim_{G(\overline{F})} g^{-1}$)

(since every automorphism of $G(F)$ is inner, and $G(F)$ has non-self dual representations)

Example

$G = SL(2,\mathbb{R})$

$$\tau(g) = xgx^{-1} \quad \left(x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

$\tau(g) \sim g^{-1}$ ($g \in$ split Cartan subgroup)

But $\tau(g) \not\sim g^{-1}$ ($g \in$ compact Cartan)

Better:

$$\tau(g) = ygy^{-1} \quad \left(y = \begin{pmatrix} i & \\ & -i \end{pmatrix}\right)$$

Then:
$C(g) = ygy^{-1}$, $C(g) \sim g^{-1}$ for all g

Moral: Focus on the fundamental (most compact) Cartan subgroup

The Real Chevalley Involution

G defined over $\mathbb{R}$, $\theta =$ Cartan involution

H is fundamental if the split rank of $H_f(\mathbb{R})$ is minimal

Example: $H_f(\mathbb{R})$ is compact

Definition

A Chevalley involution is fundamental if $C(g) = g^{-1}$ for all g in some **fundamental** Cartan subgroup of G.

The Real Chevalley Involution

Theorem

(1) There is a fundamental Chevalley involution C of G;

(2) C is defined over $\mathbb{R}$, $C : G(\mathbb{R}) \to G(\mathbb{R})$;

(3) $C(g) \sim_{G(\mathbb{R})} g^{-1}$ ($g \in G(\mathbb{R})$ semisimple)

(4) Any two fundamental Chevalley involutions are conjugate by an inner automorphism of $G(\mathbb{R})$.

Sketch of proof of the Theorem

Existence of C:

Pinning: $\mathcal{P} = (B, H, \{X_\alpha\})$

Line everything up with respect to $\mathcal{P}$

$C(X_\alpha) = X_{-\alpha}, \quad \sigma_c(X_\alpha) = -X_{-\alpha}$ (G^{σ_c} compact)

δ: distinguished automorphism (preserving $\mathcal{P}$), $\quad x \in H^\delta$

$\theta(X_\alpha) = \alpha(x) X_{\delta(\alpha)}$

$\sigma = \theta\sigma_c, \quad G(\mathbb{R}) = G^\sigma$

$$\boxed{\theta\sigma = \sigma\theta}$$

Digression

Proposition (Lusztig)

F algebraically closed $\Rightarrow$

$$C(g) \sim_G g^{-1} \text{ for all } g$$

Lemma

C = fundamental Chevalley involution

$$C(g) \sim_{G(\mathbb{R})} g^{-1} \text{ for all } g$$

(Essentially the same proof as Lusztig; thanks to Binyong Sun)

When is every π self-dual?

Since $C(g) \sim_{G(\mathbb{R})} g^{-1}$ (g semisimple):

Corollary

$$\pi \text{ irreducible } \Rightarrow \pi^C \simeq \pi^*$$

Corollary

Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if C is inner for $G(\mathbb{R})$

Necessary but not sufficient: $-1 \in W(G, H)$

When is every π self-dual?

$H_f(\mathbb{R})$ fundamental

$W(G(\mathbb{R}), H_f(\mathbb{R})) = \mathrm{Norm}_{G(\mathbb{R})}(H_f(\mathbb{R}))/H_f(\mathbb{R}) \hookrightarrow W(G, H_f)$

Proposition

Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if

$$-1 \in W(G(\mathbb{R}), H_f(\mathbb{R}))$$

(easy consequence of the Theorem)

When is every π self-dual?

G, $G(\mathbb{R}) = G\sigma$, $K = G^\theta$ (K is complex)

$H_K = H \cap K \subset H$: Cartan subgroup of K

Equal rank case: $H_K = H$

$W(K,H) \simeq W(G(\mathbb{R}), H(\mathbb{R}))$

Corollary

Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if

$$-1 \in W(K,H)$$

When is every π self-dual?

Dangerous Bend In the unequal rank case

$$W(K,H)\simeq W(K,H_K)$$

right hand side: Weyl group of a (disconnected) reductive group

but -1 has different meaning on the two sides

$x\in \mathrm{Norm}_K(H)=\mathrm{Norm}_K(H_K)$,

$$xhx^{-1}=h^{-1}\quad (h\in H_K)\not\Rightarrow xhx^{-1}=h^{-1}\quad (h\in H)$$

When is every π self-dual?

Proposition

Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if every irreducible representation of K is self-dual, and, in the unequal rank case, $-1 \in W(G,H)$

(equal rank case: $-1 \in W(K,H_K) \Rightarrow -1 \in W(G,H)$)

When is every π self-dual?

Proposition

$G(\mathbb{R})$ is simple: every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if $-1 \in W(G,H)$ and, in the equal rank case, $G(\mathbb{R})$ is a pure real form.

pure: $\theta = \mathrm{int}(x)$, $x^2 = 1$

($-1 \in W(G,H) \Rightarrow Z(G) =$ two-group $\Rightarrow$ purity independent of the choice of x) "Purity Of Essence"

Key point: $g \in \mathrm{Norm}_G(H)$ representative of $-1 \in W(G,H)$:

$$-1 \in W(K,H) \Leftrightarrow xgx^{-1} = g \Leftrightarrow x^2 g = g \Leftrightarrow x^2 = 1$$

When is every π self-dual?

Corollary

G adjoint, $-1 \in W(G,H) \Rightarrow$

$$\textit{every irreducible representation of } G(\mathbb{R}) \textit{ is self-dual}$$

List of simple $G(\mathbb{R})$, with all π self-dual

(1) A_n: $SO(2,1)$, $SU(2)$ and $SO(3)$.

(2) B_n: Every real form of the adjoint group, $Spin(2p, 2q+1)$ (p even).

(3) C_n: Every real form of the adjoint group, $Sp(p,q)$.

(4) D_{2n+1}: none.

(5) D_{2n}, unequal rank: all real forms

(6) D_{2n}, equal rank (various cases...)

(7) E_6: none.

(8) E_7: Every real form of the adjoint group, simply connected compact.

(9) G_2, F_4, E_8: every real form.

(10) complex groups of type $A_1, B_n, C_n, D_{2n}, G_2, F_4, E_7, E_8$

Frobenius-Schur indicator

Suppose $\pi \simeq \pi^*$

$T : \pi \simeq \pi^* \to \langle v, w \rangle = (Tv)(w)$

$\langle , \rangle$ bilinear, symmetric or antisymmetric:

$$\langle v, w \rangle = \epsilon_\pi \langle w, v \rangle \quad (\epsilon_\pi = \pm 1)$$

ϵ_π = Frobenius-Schur indicator

Problem

How do you compute ϵ_π?

(interesting invariant of self-dual representations)

Frobenius-Schur indicator: Finite Dimensional Representations

$G(\mathbb{R})$, $\pi \simeq \pi^*$ finite dimensional,

χ_π : central character

$$z(\rho^\vee) = \exp(2\pi i \rho^\vee) \in Z(G)$$

(fixed by all automorphisms)

Proposition (Bourbaki)

$$\epsilon_\pi = \chi_\pi(z(\rho^\vee))$$

Frobenius-Schur indicator: Finite Dimensional Representations

Key ingredient of proof:

$$w_0 \in W = W(G,H)(\text{ long element})) \to g \in \mathrm{Norm}_H(G) \quad (\text{mapping to } w_0)$$
$$\to g^2 \in H$$

Lemma

We can choose g so that

$$g^2 = z(\rho^\vee),$$

If $-1 \in W$, g^2 is independent of all choices.

(proof: uses the Tits group)

Remark: Same fact (dual side): discrete series are parametrized by $X^*(H) + \rho$

Frobenius-Schur indicator: Finite Dimensional Representations

proof of Proposition:

$$\begin{aligned}\chi_\pi(g^2)\langle v, \pi(g)v\rangle &= \langle \pi(g^2)v, \pi(g)v\rangle \\ &= \langle \pi(g)v, v\rangle \\ &= \epsilon(\pi)\langle v, \pi(g)v\rangle\end{aligned}$$

i.e.

$$\boxed{\chi_\pi(g^2)\langle v, \pi(g)v\rangle = \epsilon(\pi)\langle v, \pi(g)v\rangle}$$

Take $v \in V_\lambda$ (highest weight space), $\pi(g)v \in V_{-\lambda}$, $\langle v, \pi(g)v\rangle \neq 0$

(also see [Prasad, IMRN 1999])

Frobenius-Schur indicator

Suppose every irreducible π (infinite dimensional) is self-dual

μ : lowest K-type, multiplicity one, self-dual (by previous lemma)

$$\epsilon_\pi = \epsilon_\mu$$

Example: Assume K is connected

Take π finite dimensional

(1) $\epsilon_\pi = \chi_\pi(z(\rho_G^\vee))$ (result applied to G)

(2) $\epsilon_\pi = \epsilon_\mu = \chi_\mu(z(\rho_K^\vee))$ (result applied to K)

How can this be?

Frobenius-Schur indicator

(K connected, $-1 \in W(K,H)$)

λ=highest weight

$$\Rightarrow \lambda(z(\rho_G^\vee)) = \lambda(z(\rho_K^\vee)) \quad (\lambda \in X^*(H))$$

$$\Rightarrow z(\rho_G^\vee) = z(\rho_K^\vee)$$

Surprise:

Lemma

Assume $-1 \in W(K,H)$. *Then*

$$z(\rho_G^\vee) = z(\rho_K^\vee)$$

Frobenius-Schur indicator

$$-1 \in W(K,H) \Rightarrow z(\rho_G^\vee) = z(\rho_K^\vee):$$

Example: $G = SL(2)/PGL(2)$

$G(\mathbb{R}) = SL(2,\mathbb{R})/PGL(2,\mathbb{R}) : z(\rho_G^\vee) = -I$

$K = SO(2)/O(2)$: $z(\rho_K^\vee) = I$

$SL(2,\mathbb{R})$: $z(\rho_G^\vee) = -I \neq I = z(\rho_K^\vee)$ $(-1 \notin W(K,H))$

$PGL(2,\mathbb{R})$: $z(\rho_G^\vee) = -I = I = z(\rho_K^\vee)$ $(-1 \in W(K,H))$

Disconnectedness of K

Reduce to K^0 or $\langle K^0, C\rangle$.

Lemma

$K = \langle K^0, x_1, \ldots, x_n\rangle$ where:

(1) $x_i^2 = 1$

(2) x_i preserves a Borel of K^0

(3) x_i, x_j commute

Key point: $\mu|_{K^0}$ has multiplicity one

Frobenius-Schur indicator

Corollary

Every irreducible representation self-dual implies

$$\epsilon_\pi = \chi_\pi(z(\rho^\vee))$$

Proof of Lemma and corollary:

$z(\rho^\vee_K) = z(\rho^\vee_G)$, minimal K-type $\mu\dots$

Done if K is connected

delicate argument about the disconnectedness of K (previous slide…)

Frobenius-Schur indicator

Corollary

$-1 \in W(G,H)$, G adjoint implies every irreducible representation of $G(\mathbb{R})$ is self-dual and orthogonal.

Problem

Consider the Frobenius-Schur indicator in general

(some of the same ideas apply)