Atlas of Lie Groups and Representations

www.liegroups.org

Computing the Unitary Dual

Jeffrey Adams, David Vogan, Peter Trapa and Marc van Leeuwen

Computing the Unitary Dual

Jeffrey Adams, David Vogan, Peter Trapa and Marc van Leeuwen

Atlas web site: www.liegroups.org

Computing the Unitary Dual

Jeffrey Adams, David Vogan, Peter Trapa and Marc van Leeuwen

Atlas web site: www.liegroups.org

Web interface to the Atlas software: www.liegroups.org/web

Computing the Unitary Dual

Jeffrey Adams, David Vogan, Peter Trapa and Marc van Leeuwen

Atlas web site: www.liegroups.org

Web interface to the Atlas software: www.liegroups.org/web

Atlas Workshop: www.liegroups.org/workshop

Computing the Unitary Dual

Jeffrey Adams, David Vogan, Peter Trapa and Marc van Leeuwen

Atlas web site: www.liegroups.org

Web interface to the Atlas software: www.liegroups.org/web

Atlas Workshop: www.liegroups.org/workshop

References: Algorithms for Representation of Real Groups
Examples of the Atlas software…
www.liegroups.org/papers

Computing the Unitary Dual

Jeffrey Adams, David Vogan, Peter Trapa and Marc van Leeuwen

Atlas web site: www.liegroups.org

Web interface to the Atlas software: www.liegroups.org/web

Atlas Workshop: www.liegroups.org/workshop

References: Algorithms for Representation of Real Groups
Examples of the Atlas software…
www.liegroups.org/papers

These slides: www.liegroups.org/talks

Atlas Project Members

- Jeffrey Adams
- Dan Barbasch
- Birne Binegar
- Bill Casselman
- Dan Ciubotaru
- Scott Crofts
- Fokko du Cloux
- Tatiana Howard
- Steven Jackson
- Alfred Noel
- Alessandra Pantano
- Annegret Paul
- Patrick Polo
- Siddhartha Sahi
- Susana Salamanca
- John Stembridge
- Peter Trapa
- Marc van Leeuwen
- David Vogan
- Wai-Ling Yee
- Jiu-Kang Yu
- Gregg Zuckerman

Atlas Project Members, AIM, July 2007

Problem: Classify the irreducible unitary representations of a real reductive group G (such as $GL(n,\mathbb{R})$, $U(p,q)$, $Sp(2n,\mathbb{R})$, E_8)

Problem: Classify the irreducible unitary representations of a real reductive group G (such as $GL(n,\mathbb{R}), U(p,q), Sp(2n,\mathbb{R}), E_8$)

$\widehat{G}_u$ = unitary dual (irreducible representations equipped with a positive definite invariant Hermitian form)

Problem: Classify the irreducible unitary representations of a real reductive group G (such as $GL(n,\mathbb{R})$, $U(p,q)$, $Sp(2n,\mathbb{R})$, E_8)

$\widehat{G}_u$ = unitary dual (irreducible representations equipped with a positive definite invariant Hermitian form)

$\widehat{G}_h$: Hermitian dual (invariant Hermitian form, not necessarily positive definite)

Problem: Classify the irreducible unitary representations of a real reductive group G (such as $GL(n,\mathbb{R})$, $U(p,q)$, $Sp(2n,\mathbb{R})$, E_8)

$\widehat{G}_u$ = unitary dual (irreducible representations equipped with a positive definite invariant Hermitian form)

$\widehat{G}_h$: Hermitian dual (invariant Hermitian form, not necessarily positive definite)

$$\widehat{G}_u \subset \widehat{G}_h$$

Other duals

$\widehat{G}_d$: discrete series

$\widehat{G}_t$: tempered dual (Plancherel measure)

$\widehat{G}_a$: admissible dual

$\widehat{G}_h$: Hermitian dual

$\widehat{G}_u$ = unitary dual

Other duals

$\widehat{G}_d$: discrete series *(Harish-Chandra)*

$\widehat{G}_t$: tempered dual (Plancherel measure) *(Harish-Chandra)*

$\widehat{G}_a$: admissible dual *(Langlands/Knapp-Zuckerman/Vogan)*

$\widehat{G}_h$: Hermitian dual *(Knapp/Zuckerman)*

$\widehat{G}_u$ = unitary dual

Other duals

$\widehat{G}_d$: discrete series *(Harish-Chandra)*

$\widehat{G}_t$: tempered dual (Plancherel measure) *(Harish-Chandra)*

$\widehat{G}_a$: admissible dual *(Langlands/Knapp-Zuckerman/Vogan)*

$\widehat{G}_h$: Hermitian dual *(Knapp/Zuckerman)*

$\widehat{G}_u$ = unitary dual

$$\boxed{\widehat{G}_d \subset \widehat{G}_t \subset \widehat{G}_u \subset \widehat{G}_h \subset \widehat{G}_a}$$

ADMISSIBLE DUAL

First step: compute the admissible dual (Atlas project, Fokko du Cloux)
For example, consider the irreducible representations of $Sp(4, \mathbb{R})$:

First step: compute the admissible dual (Atlas project, Fokko du Cloux)
For example, consider the irreducible representations of $Sp(4,\mathbb{R})$:

```
 0( 0,6): 0 0 [i1,i1] 1  2  ( 6, *) ( 4, *)
 1( 1,6): 0 0 [i1,i1] 0  3  ( 6, *) ( 5, *)
 2( 2,6): 0 0 [ic,i1] 2  0  ( *, *) ( 4, *)
 3( 3,6): 0 0 [ic,i1] 3  1  ( *, *) ( 5, *)
 4( 4,4): 1 2 [C+,r1] 8  4  ( *, *) ( 0, 2) 2
 5( 5,4): 1 2 [C+,r1] 9  5  ( *, *) ( 1, 3) 2
 6( 6,5): 1 1 [r1,C+] 6  7  ( 0, 1) ( *, *) 1
 7( 7,2): 2 1 [i2,C-] 7  6  (10,11) ( *, *) 2,1,2
 8( 8,3): 2 2 [C-,i1] 4  9  ( *, *) (10, *) 1,2,1
 9( 9,3): 2 2 [C-,i1] 5  8  ( *, *) (10, *) 1,2,1
10(10,0): 3 3 [r2,r1] 11 10 ( 7, *) ( 8, 9) 1,2,1,2
11(10,1): 3 3 [r2,rn] 10 11 ( 7, *) ( *, *) 1,2,1,2
```

Change of notation: $G = G(\mathbb{C})$, connected reductive

Change of notation: $G = G(\mathbb{C})$, connected reductive

$G(\mathbb{R})$ is a real form of G, K is a complexified maximal compact subgroup of $G(\mathbb{R})$

Change of notation: $G = G(\mathbb{C})$, connected reductive

$G(\mathbb{R})$ is a real form of G, K is a complexified maximal compact subgroup of $G(\mathbb{R})$

Representations of $G(\mathbb{R}) \to (\mathfrak{g}, K)$-modules

Change of notation: $G = G(\mathbb{C})$, connected reductive

$G(\mathbb{R})$ is a real form of G, K is a complexified maximal compact subgroup of $G(\mathbb{R})$

Representations of $G(\mathbb{R}) \to (\mathfrak{g}, K)$-modules

$G = G(\mathbb{C})$, B = Borel subgroup, θ=involution of G, $K = G^\theta$

Change of notation: $G = G(\mathbb{C})$, connected reductive

$G(\mathbb{R})$ is a real form of G, K is a complexified maximal compact subgroup of $G(\mathbb{R})$

Representations of $G(\mathbb{R}) \to (\mathfrak{g}, K)$-modules

$G = G(\mathbb{C})$, B = Borel subgroup, θ=involution of G, $K = G^\theta$

$G^\vee$ = dual group, $B^\vee$, $K^\vee$ similarly

Assume G is simple, simply connected and adjoint (G_2, F_4 or E_8)

Assume G is simple, simply connected and adjoint (G_2, F_4 or E_8)

Theorem: The irreducible representations of real forms of G, with trivial infinitesimal character, are parametrized by pairs

$$(\mathcal{O}, \mathcal{O}^\vee)$$

where:

Assume G is simple, simply connected and adjoint (G_2, F_4 or E_8)

Theorem: The irreducible representations of real forms of G, with trivial infinitesimal character, are parametrized by pairs

$$(\mathcal{O}, \mathcal{O}^\vee)$$

where:

$\mathcal{O}$ is a K-orbit on G/B (some K)

Assume G is simple, simply connected and adjoint (G_2, F_4 or E_8)

Theorem: The irreducible representations of real forms of G, with trivial infinitesimal character, are parametrized by pairs

$$(\mathcal{O}, \mathcal{O}^\vee)$$

where:

$\mathcal{O}$ is a K-orbit on G/B (some K)

$\mathcal{O}^\vee$ is a $K^\vee$-orbit on $G^\vee/B^\vee$ (some $K^\vee$)
(satisfying a certain elementary condition)

Parametrizing the Admissible Dual

Assume G is simple, simply connected and adjoint (G_2, F_4 or E_8)

Theorem: The irreducible representations of real forms of G, with trivial infinitesimal character, are parametrized by pairs

$$(\mathcal{O}, \mathcal{O}^\vee)$$

where:

$\mathcal{O}$ is a K-orbit on G/B (some K)

$\mathcal{O}^\vee$ is a $K^\vee$-orbit on $G^\vee/B^\vee$ (some $K^\vee$)
(satisfying a certain elementary condition)

[via: ${}^\vee O \rightarrow$ local system on $\mathcal{O} \rightarrow \pi$ by $\mathcal{D}$-modules…]

Assume G is simple, simply connected and adjoint (G_2, F_4 or E_8)

Theorem: The irreducible representations of real forms of G, with trivial infinitesimal character, are parametrized by pairs

$$(\mathcal{O}, \mathcal{O}^\vee)$$

where:

$\mathcal{O}$ is a K-orbit on G/B (some K)

$\mathcal{O}^\vee$ is a $K^\vee$-orbit on $G^\vee/B^\vee$ (some $K^\vee$)
(satisfying a certain elementary condition)

[via: ${}^\vee O \to$ local system on $\mathcal{O} \to \pi$ by $\mathcal{D}$-modules…]

(General G: need "strong" real forms and other infinitesimal characters)

So: compute explicit combinatorial sets parametrizing:

So: compute explicit combinatorial sets parametrizing:

(1) real forms of G (involutions θ, K)

So: compute explicit combinatorial sets parametrizing:

(1) real forms of G (involutions θ, K)

(2) $K\backslash G/B$ (K-orbits on the flag variety): $x = 0, \ldots, m$

So: compute explicit combinatorial sets parametrizing:

(1) real forms of G (involutions θ, K)

(2) $K\backslash G/B$ (K-orbits on the flag variety): $x = 0, \ldots, m$

(3) $G^\vee$, $K^\vee$, $K^\vee\backslash G^\vee/B^\vee$: $y = 0, \ldots, n$

So: compute explicit combinatorial sets parametrizing:

(1) real forms of G (involutions θ, K)

(2) $K\backslash G/B$ (K-orbits on the flag variety): $x = 0, \dots, m$

(3) $G^\vee, K^\vee, K^\vee\backslash G^\vee/B^\vee$: $y = 0, \dots, n$

$(K, K^\vee) \to$ Block of representations parametrized by (subset of pairs) (x, y)

So: compute explicit combinatorial sets parametrizing:

(1) real forms of G (involutions θ, K)

(2) $K\backslash G/B$ (K-orbits on the flag variety): $x = 0, \dots, m$

(3) $G^\vee$, $K^\vee$, $K^\vee\backslash G^\vee/B^\vee$: $y = 0, \dots, n$

$(K, K^\vee) \to$ Block of representations parametrized by (subset of pairs) (x, y)

This is (some of) what the Atlas software does.

Block of $Sp(4,\mathbb{R})$:

```
 0( 0,6): 0 0 [i1,i1] 1  2  ( 6, *) ( 4, *)
 1( 1,6): 0 0 [i1,i1] 0  3  ( 6, *) ( 5, *)
 2( 2,6): 0 0 [ic,i1] 2  0  ( *, *) ( 4, *)
 3( 3,6): 0 0 [ic,i1] 3  1  ( *, *) ( 5, *)
 4( 4,4): 1 2 [C+,r1] 8  4  ( *, *) ( 0, 2) 2
 5( 5,4): 1 2 [C+,r1] 9  5  ( *, *) ( 1, 3) 2
 6( 6,5): 1 1 [r1,C+] 6  7  ( 0, 1) ( *, *) 1
 7( 7,2): 2 1 [i2,C-] 7  6  (10,11) ( *, *) 2,1,2
 8( 8,3): 2 2 [C-,i1] 4  9  ( *, *) (10, *) 1,2,1
 9( 9,3): 2 2 [C-,i1] 5  8  ( *, *) (10, *) 1,2,1
10(10,0): 3 3 [r2,r1] 11 10 ( 7, *) ( 8, 9) 1,2,1,2
11(10,1): 3 3 [r2,rn] 10 11 ( 7, *) ( *, *) 1,2,1,2
```

$(\pi, V) = \mathfrak{g}_0$ module ($\mathfrak{g}_0$ = real Lie algebra)

Hermitian Dual

(π, V) = $\mathfrak{g}_0$ module ($\mathfrak{g}_0$ = real Lie algebra)

Definition: A Hermitian form $\langle\ ,\ \rangle$ on V is invariant if

$$\langle \pi(X)v, w\rangle + \langle v, \pi(X)w\rangle = 0 \quad (X \in \mathfrak{g}_0)$$

$(\pi, V) = \mathfrak{g}_0$ module ($\mathfrak{g}_0$ = real Lie algebra)

Definition: A Hermitian form $\langle\ ,\ \rangle$ on V is invariant if

$$\langle \pi(X)v, w\rangle + \langle v, \pi(X)w\rangle = 0 \quad (X \in \mathfrak{g}_0)$$

i.e.

$$\langle \pi(X)v, w\rangle + \langle v, \pi(\overline{X})w\rangle = 0 \quad (X \in \mathfrak{g})$$

$(\pi, V) = \mathfrak{g}_0$ module ($\mathfrak{g}_0$ = real Lie algebra)

Definition: A Hermitian form $\langle\ ,\ \rangle$ on V is invariant if

$$\langle \pi(X)v, w\rangle + \langle v, \pi(X)w\rangle = 0 \quad (X \in \mathfrak{g}_0)$$

i.e.

$$\langle \pi(X)v, w\rangle + \langle v, \pi(\overline{X})w\rangle = 0 \quad (X \in \mathfrak{g})$$

$(\pi, V) = (\mathfrak{g}, K)$-module, $G(\mathbb{R})$ corresponding real group, σ an antiholomorphic involution of $G = G(\mathbb{C})$, $G(\mathbb{R}) = G(\mathbb{C})^\sigma$

(π, V) = $\mathfrak{g}_0$ module ($\mathfrak{g}_0$ = real Lie algebra)

Definition: A Hermitian form $\langle\ ,\ \rangle$ on V is invariant if

$$\langle \pi(X)v, w\rangle + \langle v, \pi(X)w\rangle = 0 \quad (X \in \mathfrak{g}_0)$$

i.e.

$$\langle \pi(X)v, w\rangle + \langle v, \pi(\overline{X})w\rangle = 0 \quad (X \in \mathfrak{g})$$

(π, V) = $(\mathfrak{g}, K)$-module, $G(\mathbb{R})$ corresponding real group, σ an antiholomorphic involution of $G = G(\mathbb{C})$, $G(\mathbb{R}) = G(\mathbb{C})^\sigma$

Definition: A Hermitian form $\langle\ ,\ \rangle$ on (π, V) is invariant if

$$\langle \pi(X)v, w\rangle + \langle v, \pi(\sigma(X))w\rangle = 0 \quad (X \in \mathfrak{g})$$

(and a similar condition for K)

Definition (Knapp/Zuckerman):

$$V^h = \{f : V \to \mathbb{C} \mid f(\lambda v) = \overline{\lambda} f(v)\}$$

(K-finite functions)

Hermitian Dual

Definition (Knapp/Zuckerman):

$$V^h = \{f : V \to \mathbb{C} \mid f(\lambda v) = \overline{\lambda} f(v)\}$$

(K-finite functions)

σ as before ($G(\mathbb{R}) = G^\sigma$)

Definition (Knapp/Zuckerman):

$$V^h = \{f : V \to \mathbb{C} \mid f(\lambda v) = \overline{\lambda} f(v)\}$$

(K-finite functions)

σ as before ($G(\mathbb{R}) = G^\sigma$)

Definition: The Hermitian Dual of (π, V) is the $(\mathfrak{g}, K)$-module (π_σ^h, V^h):

Definition (Knapp/Zuckerman):

$$V^h = \{f : V \to \mathbb{C} \mid f(\lambda v) = \overline{\lambda} f(v)\}$$

(K-finite functions)

σ as before ($G(\mathbb{R}) = G^\sigma$)

Definition: The Hermitian Dual of (π, V) is the $(\mathfrak{g}, K)$-module (π^h_σ, V^h):

$$\pi^h_\sigma(X)(f)(v) = -f(\sigma(x)v)$$

$$\pi^h_\sigma(k)(f)(v) = f(\sigma(k)^{-1}v)$$

Definition (Knapp/Zuckerman):

$$V^h = \{f : V \to \mathbb{C} \mid f(\lambda v) = \overline{\lambda} f(v)\}$$

(K-finite functions)

σ as before ($G(\mathbb{R}) = G^\sigma$)

Definition: The Hermitian Dual of (π, V) is the $(\mathfrak{g}, K)$-module (π_σ^h, V^h):

$$\pi_\sigma^h(X)(f)(v) = -f(\sigma(x)v)$$

$$\pi_\sigma^h(k)(f)(v) = f(\sigma(k)^{-1}v)$$

(σ exponentiated to K)

Proposition:
An invariant Hermitian form on (π, V) is the same thing as an intertwining operator $T : V \to V^h$

Proposition:
An invariant Hermitian form on (π, V) is the same thing as an intertwining operator $T : V \to V^h$

If (π, V) is irreducible then $(\pi, V) \simeq (\pi^h_\sigma, V^h)$ if and only if (π, V) has an invariant Hermitian form.

Proposition:
An invariant Hermitian form on (π, V) is the same thing as an intertwining operator $T : V \to V^h$

If (π, V) is irreducible then $(\pi, V) \simeq (\pi^h_\sigma, V^h)$ if and only if (π, V) has an invariant Hermitian form.

$\phi : V \to V^h \longrightarrow \langle v, w\rangle = \lambda\phi(v)(w)$

Proposition:
An invariant Hermitian form on (π, V) is the same thing as an intertwining operator $T : V \to V^h$

If (π, V) is irreducible then $(\pi, V) \simeq (\pi^h_\sigma, V^h)$ if and only if (π, V) has an invariant Hermitian form.

$\phi : V \to V^h \longrightarrow \langle v, w\rangle = \lambda\phi(v)(w)$

[$|\lambda| = 1$ to turn sesquilinear into Hermitian]

Proposition:
An invariant Hermitian form on (π, V) is the same thing as an intertwining operator $T : V \to V^h$

If (π, V) is irreducible then $(\pi, V) \simeq (\pi^h_\sigma, V^h)$ if and only if (π, V) has an invariant Hermitian form.

$\phi : V \to V^h \longrightarrow \langle v, w\rangle = \lambda\phi(v)(w)$

[$|\lambda| = 1$ to turn sesquilinear into Hermitian]

Shorthand: $\pi \to \pi^h$ (Hermitian dual operation): involution of the admissible dual

Proposition:
An invariant Hermitian form on (π, V) is the same thing as an intertwining operator $T : V \to V^h$

If (π, V) is irreducible then $(\pi, V) \simeq (\pi^h_\sigma, V^h)$ if and only if (π, V) has an invariant Hermitian form.

$\phi : V \to V^h \longrightarrow \langle v, w\rangle = \lambda\phi(v)(w)$

$[|\lambda| = 1$ to turn sesquilinear into Hermitian$]$

Shorthand: $\pi \to \pi^h$ (Hermitian dual operation): involution of the admissible dual

Problem: Compute $\pi \to \pi^h$ in atlas parameters

Digression: c-invariant forms

Recall:
(π_σ^h, V^h):

$$\pi_\sigma^h(X)(f)(v) = -f(\sigma(x)v)$$

$$\pi_\sigma^h(k)(f)(v) = f(\sigma(k)^{-1}v)$$

Recall:
(π_σ^h, V^h):

$$\pi_\sigma^h(X)(f)(v) = -f(\sigma(x)v)$$

$$\pi_\sigma^h(k)(f)(v) = f(\sigma(k)^{-1}v)$$

We could use any anti-holomorphic involution (preserving K). For example σ_c, corresponding to a compact real form of G.

Digression: c-invariant forms

Recall:
(π^h_σ, V^h):

$$\pi^h_\sigma(X)(f)(v) = -f(\sigma(x)v)$$

$$\pi^h_\sigma(k)(f)(v) = f(\sigma(k)^{-1}v)$$

We could use any anti-holomorphic involution (preserving K). For example σ_c, corresponding to a compact real form of G.

Definition: The σ_c-Hermitian Dual of (π, V) is the $(\mathfrak{g}, K)$-module $(\pi^h_{\sigma_c}, V^h)$:

$$\pi^h_{\sigma_c}(X)(f)(v) = -f(\sigma_c(x)v)$$

$$\pi^h_{\sigma_c}(k)(f)(v) = f(\sigma_c(k)^{-1}v)$$

This defines a σ_c-invariant Hermitian form $\langle\,,\,\rangle_c$:

$$\langle\pi(X)v, w\rangle_c + \langle v, \pi(\sigma_c(X))w\rangle_c = 0$$

DIGRESSION: C-INVARIANT FORMS

This defines a σ_c-invariant Hermitian form $\langle\ ,\ \rangle_c$:

$$\langle \pi(X)v, w\rangle_c + \langle v, \pi(\sigma_c(X))w\rangle_c = 0$$

Note: The σ and σ_c invariant forms differ by something (essentially) inner and easy to compute.

Digression: c-invariant forms

This defines a σ_c-invariant Hermitian form $\langle\ ,\ \rangle_c$:

$$\langle \pi(X)v, w\rangle_c + \langle v, \pi(\sigma_c(X))w\rangle_c = 0$$

Note: The σ and σ_c invariant forms differ by something (essentially) inner and easy to compute.

The c-invariant (shorthand for σ_c-invariant) forms have a lot of advantages (later. . .)

Action of Hermitan Dual on Parameters

Atlas setup: an inner class of real forms of G

Action of Hermitan Dual on Parameters

Atlas setup: an inner class of real forms of G

Given by an outer automorphism τ of G; (Example: $\tau = 1 \leftrightarrow$ real forms containing a compact Cartan subgroup)

Action of Hermitan Dual on Parameters

Atlas setup: an inner class of real forms of G

Given by an outer automorphism τ of G; (Example: $\tau = 1 \leftrightarrow$ real forms containing a compact Cartan subgroup)

Example: $G = GL(n, \mathbb{C})$

Action of Hermitan Dual on Parameters

Atlas setup: an inner class of real forms of G

Given by an outer automorphism τ of G; (Example: $\tau = 1 \leftrightarrow$ real forms containing a compact Cartan subgroup)

Example: $G = GL(n, \mathbb{C})$

$\tau = 1$: $\{U(p,q) \mid p+q = n\}$

ACTION OF HERMITAN DUAL ON PARAMETERS

Atlas setup: an inner class of real forms of G

Given by an outer automorphism τ of G; (Example: $\tau = 1 \leftrightarrow$ real forms containing a compact Cartan subgroup)

Example: $G = GL(n, \mathbb{C})$

$\tau = 1$: $\{U(p,q) \mid p+q=n\}$

$\tau(g) = {}^t g^{-1}$: $GL(n,\mathbb{R})$ (and $GL(n/2,\mathbb{H})$)

Action of Hermitan Dual on Parameters

Atlas setup: an inner class of real forms of G

Given by an outer automorphism τ of G; (Example: $\tau = 1 \leftrightarrow$ real forms containing a compact Cartan subgroup)

Example: $G = GL(n, \mathbb{C})$

$\tau = 1$: $\{U(p, q) \mid p + q = n\}$

$\tau(g) = {}^t g^{-1}$: $GL(n, \mathbb{R})$ (and $GL(n/2, \mathbb{H})$)

τ acts on G/B, $K\backslash G/B$

Action of Hermitan Dual on Parameters

Atlas setup: an inner class of real forms of G

Given by an outer automorphism τ of G; (Example: $\tau = 1 \leftrightarrow$ real forms containing a compact Cartan subgroup)

Example: $G = GL(n, \mathbb{C})$

$\tau = 1$: $\{U(p,q) \mid p+q=n\}$

$\tau(g) = {}^t g^{-1}$: $GL(n,\mathbb{R})$ (and $GL(n/2,\mathbb{H})$)

τ acts on G/B, $K\backslash G/B$

natural "transpose"

τ^t on $K^\vee\backslash G^\vee/B^\vee$

Recall representations are parametrized by a $K \times K^\vee$-orbit

$$(\mathcal{O}, \mathcal{O}^\vee) \subset G/B \times G^\vee/B^\vee$$

Action of Hermitan Dual on Parameters

Recall representations are parametrized by a $K \times K^\vee$-orbit

$$(\mathcal{O}, \mathcal{O}^\vee) \subset G/B \times G^\vee/B^\vee$$

Assume π has real infinitesimal character (example: discrete series, principal series induced from a real valued character of A):

Action of Hermitan Dual on Parameters

Recall representations are parametrized by a $K \times K^\vee$-orbit

$$(\mathcal{O}, \mathcal{O}^\vee) \subset G/B \times G^\vee/B^\vee$$

Assume π has real infinitesimal character (example: discrete series, principal series induced from a real valued character of A):

Proposition: Suppose

$$\pi \leftrightarrow (\mathcal{O}, \mathcal{O}^\vee)$$

Then

Action of Hermitan Dual on Parameters

Recall representations are parametrized by a $K \times K^\vee$-orbit

$$(\mathcal{O}, \mathcal{O}^\vee) \subset G/B \times G^\vee/B^\vee$$

Assume π has real infinitesimal character (example: discrete series, principal series induced from a real valued character of A):

Proposition: Suppose

$$\pi \leftrightarrow (\mathcal{O}, \mathcal{O}^\vee)$$

Then

$$\pi^h \leftrightarrow (\tau(\mathcal{O}), \tau^t(\mathcal{O}^\vee))$$

Action of Hermitan Dual on Parameters

Recall representations are parametrized by a $K \times K^\vee$-orbit

$$(\mathcal{O}, \mathcal{O}^\vee) \subset G/B \times G^\vee/B^\vee$$

Assume π has real infinitesimal character (example: discrete series, principal series induced from a real valued character of A):

Proposition: Suppose

$$\pi \leftrightarrow (\mathcal{O}, \mathcal{O}^\vee)$$

Then

$$\pi^h \leftrightarrow (\tau(\mathcal{O}), \tau^t(\mathcal{O}^\vee))$$

(elementary to compute)

Action of Hermitan Dual on Parameters

Recall representations are parametrized by a $K \times K^\vee$-orbit

$$(\mathcal{O}, \mathcal{O}^\vee) \subset G/B \times G^\vee/B^\vee$$

Assume π has real infinitesimal character (example: discrete series, principal series induced from a real valued character of A):

Proposition: Suppose

$$\pi \leftrightarrow (\mathcal{O}, \mathcal{O}^\vee)$$

Then

$$\pi^h \leftrightarrow (\tau(\mathcal{O}), \tau^t(\mathcal{O}^\vee))$$

(elementary to compute)

Corollary: If G is equal rank every irreducible representation (with real infinitesimal character) is Hermitian.

Basic idea: Kazhdan-Lusztig-Vogan theory expresses irreducible representation as formal sums of standard modules

Basic idea: Kazhdan-Lusztig-Vogan theory expresses irreducible representation as formal sums of standard modules

Program: Develop Kazhdan-Lusztig-Vogan theory for representations with Hermitian forms

Basic idea: Kazhdan-Lusztig-Vogan theory expresses irreducible representation as formal sums of standard modules

Program: Develop Kazhdan-Lusztig-Vogan theory for representations with Hermitian forms

(1) Express irreducible $(\pi, \langle\ ,\ \rangle)$ as a formal of standard modules equipped with Hermitian forms

Basic idea: Kazhdan-Lusztig-Vogan theory expresses irreducible representation as formal sums of standard modules

Program: Develop Kazhdan-Lusztig-Vogan theory for representations with Hermitian forms

(1) Express irreducible $(\pi, \langle\ ,\ \rangle)$ as a formal of standard modules equipped with Hermitian forms

(2) Compute Hermitian forms on standard modules

Program to Compute Hermitian Forms

Basic idea: Kazhdan-Lusztig-Vogan theory expresses irreducible representation as formal sums of standard modules

Program: Develop Kazhdan-Lusztig-Vogan theory for representations with Hermitian forms

(1) Express irreducible $(\pi, \langle\ ,\ \rangle)$ as a formal of standard modules equipped with Hermitian forms

(2) Compute Hermitian forms on standard modules

(3) 1+2 $\Rightarrow$ computation of Hermitian forms on irreducible representations

Program to Compute Hermitian Forms

Basic idea: Kazhdan-Lusztig-Vogan theory expresses irreducible representation as formal sums of standard modules

Program: Develop Kazhdan-Lusztig-Vogan theory for representations with Hermitian forms

(1) Express irreducible $(\pi, \langle\ ,\ \rangle)$ as a formal of standard modules equipped with Hermitian forms

(2) Compute Hermitian forms on standard modules

(3) 1+2 $\Rightarrow$ computation of Hermitian forms on irreducible representations

(4) For each irreducible Hermitian representation, check if its Hermitian form is positive definite

Recall:

Verma modules:

Recall:
Verma modules:

Parameter set: W
$M(w)$: Verma module with highest weight $-w\rho - \rho$
$L(w)$: unique irreducible quotient of $M(w)$

Recall:

Verma modules:

Parameter set: W

$M(w)$: Verma module with highest weight $-w\rho - \rho$

$L(w)$: unique irreducible quotient of $M(w)$

$(\mathfrak{g}, K)$-modules:

Parameter set: $\{\gamma = (x, y)\}$ (pair of orbits)

$I(\gamma)$ = standard module (full induced from discrete series)

$J(\gamma)$ = unique irreducible quotient of $I(\gamma)$

Verma modules

$$M(w) = \sum_{v \leq w} m(v,w) L(v) \quad (m(v,w) \in \mathbb{Z} \geq 0)$$

$$L(w) = \sum_{v \leq w} M(v,w) M(v) \quad (M(v,w) = \pm P_{v,w}(1) \in \mathbb{Z})$$

Verma modules

$$M(w) = \sum_{v \leq w} m(v,w) L(v) \quad (m(v,w) \in \mathbb{Z} \geq 0)$$

$$L(w) = \sum_{v \leq w} M(v,w) M(v) \quad (M(v,w) = \pm P_{v,w}(1) \in \mathbb{Z})$$

$P_{v,w}$ = Kazhdan-Lusztig polynomial (for $B \backslash G / B$)

Verma modules

$$M(w) = \sum_{v \le w} m(v,w) L(v) \quad (m(v,w) \in \mathbb{Z} \ge 0)$$

$$L(w) = \sum_{v \le w} M(v,w) M(v) \quad (M(v,w) = \pm P_{v,w}(1) \in \mathbb{Z})$$

$P_{v,w}$ = Kazhdan-Lusztig polynomial (for $B \backslash G / B$)

Harish-Chandra modules (working in a fixed block):

$$I(\gamma) = \sum_{\delta \le \gamma} m(\delta,\gamma) J(\delta) \quad (m(\delta,\gamma) \in \mathbb{Z} \ge 0)$$

$$J(\gamma) = \sum_{\delta \le \gamma} M(\delta,\gamma) I(\delta) \quad (M(\delta,\gamma) = \pm P_{\gamma,\delta}(1) \in \mathbb{Z})$$

KLV Polynomials

Verma modules

$$M(w)=\sum_{v\le w} m(v,w)L(v)\quad (m(v,w)\in\mathbb{Z}\ge 0)$$
$$L(w)=\sum_{v\le w} M(v,w)M(v)\quad (M(v,w)=\pm P_{v,w}(1)\in\mathbb{Z})$$

$P_{v,w}$ = Kazhdan-Lusztig polynomial (for $B\backslash G/B$)

Harish-Chandra modules (working in a fixed block):

$$I(\gamma)=\sum_{\delta\le\gamma} m(\delta,\gamma)J(\delta)\quad (m(\delta,\gamma)\in\mathbb{Z}\ge 0)$$
$$J(\gamma)=\sum_{\delta\le\gamma} M(\delta,\gamma)I(\delta)\quad (M(\delta,\gamma)=\pm P_{\gamma,\delta}(1)\in\mathbb{Z})$$

$P_{\delta,\gamma}$ = Kazhdan-Lusztig-Vogan polynomial (for $K\backslash G/B$), (computed by Atlas software)

Goal:

Find formulas for the form:

$$\langle\, ,\, \rangle_{J(\gamma)} = \sum_{\delta \leq \gamma} H_{\gamma,\delta} \langle\, ,\, \rangle_{I(\gamma)}$$

Goal:

Find formulas for the form:

$$\langle\, ,\, \rangle_{J(\gamma)} = \sum_{\delta \leq \gamma} H_{\gamma,\delta} \langle\, ,\, \rangle_{I(\gamma)}$$

This is a refinement of:

$$J(\gamma) = \sum_{\delta \leq \gamma} M(\delta, \gamma) I(\delta)$$

Goal:

Find formulas for the form:

$$\langle\, ,\, \rangle_{J(\gamma)} = \sum_{\delta\le\gamma} H_{\gamma,\delta} \langle\, ,\, \rangle_{I(\gamma)}$$

This is a refinement of:

$$J(\gamma) = \sum_{\delta\le\gamma} M(\delta,\gamma) I(\delta)$$

in the language of signature characters:

$$J(\gamma) = (\sum_{\delta\le\gamma} H^+_{\delta,\gamma} I(\delta), \sum_{\delta\le\gamma} H^-_{\delta,\gamma} I(\delta))$$

$$H^+_{\delta,\gamma} + H^-_{\delta,\gamma} = M(\delta,\gamma)$$

Problems:

Problems:

(1) $I(\gamma)$ may not be Hermitian (in unequal rank case)

Problems:

(1) $I(\gamma)$ may not be Hermitian (in unequal rank case)

(2) No canonical choice of Hermitian forms $\langle\, ,\, \rangle_{I(\gamma)}$, $\langle\, ,\, \rangle_{J(\gamma)}$.

Problems:

(1) $I(\gamma)$ may not be Hermitian (in unequal rank case)

(2) No canonical choice of Hermitian forms $\langle\,,\,\rangle_{I(\gamma)}, \langle\,,\,\rangle_{J(\gamma)}$.

BIG IDEA: These go away if you use c-invariant forms: these always exist and can be made canonical.

Problems:

(1) $I(\gamma)$ may not be Hermitian (in unequal rank case)

(2) No canonical choice of Hermitian forms $\langle\, ,\, \rangle_{I(\gamma)}, \langle\, ,\, \rangle_{J(\gamma)}$.

BIG IDEA: These go away if you use c-invariant forms: these always exist and can be made canonical.

Continued. . .