Closure Diagrams for Nilpotent Orbits of Exceptional Groups

$G$: simple, connected, complex exceptional group
$O$: a nilpotent orbit of $G$ on Lie$(G)$;
$\mathcal{N}(G)$: the (finite) set of nilpotent orbits;
$G^\vee$: the dual group;
$d: \mathcal{N}(G) \to \mathcal{N}(G^\vee)$: order-reversing duality of nilpotent orbits;

Further terminology is explained below. Each page has a graph of the partially ordered set $\mathcal{N}(G)$:

**Nodes:**
Each node corresponds to a nilpotent orbit $O$
- **Dark Blue:** $O$ is even ($\Rightarrow$ special)
- **Light Blue:** $O$ is special, but not even
- **White:** $O$ is not special
- **Red border:** $d(O)$ is even

**Edges:**
Edges are closure relations: $O_2 \subseteq O_1$
- **Green edge:** $O_1, O_2$ are in the same special piece

Further information and terminology:

- $O \in \mathcal{N}(G)$ is special if $O = d(O^\vee)$ for some $O^\vee \in \mathcal{N}(G^\vee)$;
- $\mathcal{N}_s(G)$: the special orbits; $d$ is a bijection between $\mathcal{N}_s(G)$ and $\mathcal{N}_s(G^\vee)$;
- the special piece of $O$ is the set of orbits $O'$ satisfying $d(O) = d(O')$; this consists of a single special orbit $O_s$, together with non-special orbits in the closure of $O_s$, but not in the closure of any smaller special orbit
- $A(O) = \text{Cent}_G(X)/\text{Cent}_G(X)^0$ ($X \in O$);
- $\overline{A}(O)$ (for $O$ special): Lusztig’s canonical quotient of $A(O)$ (cf. [7]), $\overline{A}(O) \simeq \overline{A}(d(O))$;

The last line of each node is: $\overline{A}(O)$ (if $O$ is special), followed by $(A(O), A(d(O)))$.

D: distinguished (intersects no proper Levi)
R: rigid (not induced)
bi-R: birationally rigid (not birationally induced, i.e. the corresponding moment map is birational, cf. [4]);
q-D: quasi-distinguished ([10]): a unipotent element $u$ is quasi-distinguished if there is a semisimple element $t$ so that $tu = ut$ and $u$ is distinguished in the centralizer of $t$. Quasi-distinguished implies distinguished (take $t = 1$), and implies that the reductive part of the centralizer of $u$ is a torus.

Some implications which are known:

- **distinguished $\Rightarrow$ even $\Rightarrow$ special (cf. [6, Theorem 8.2.3])
- **even $\Rightarrow$ birationally induced from (the 0-orbit on) a proper Levi ([1, Lemma 27.8])
• quasi-distinguished ⇒ even except for two cases in $E_8$ (case-by-case)

**Nota Bene:** I am taking the pictures in [12] as being authoritative. These are reproduced in [5]; however there are some mistakes in the latter. Earlier references for $E_6$, $E_7$ and $E_8$ are [8] and [9]. These pictures also have a few errors. The pictures are determined by the Green functions, which are computed in [3]. Also see [2], [6], and [11].

There are two versions of the $E_8$ pictures; which one you use depends on whether you are less than 50 years old or not.

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**References**


$G_2$

- $G_2 [D]$
  - 0 [R]
  - 1 (1/1)
- $G_2(a1) [D]$
  - $G_2(a1) [D]$
  - S3 (S3/S3)
- A1s [R]
  - $G_2(a1) [D]$
  - (1/S3)
- A1 [R]
  - $G_2(a1) [D]$
  - (1/S3)
- 0 [R]
  - $G_2 [D]$
  - 1 (1/1)
$E_8$ I