Closure Diagrams for Nilpotent Orbits of Exceptional Groups

G: simple, connected, complex exceptional group
$\mathcal{O}$: a nilpotent orbit of G on Lie(G);
$\mathcal{N}(G)$: the (finite) set of nilpotent orbits;
$G^\vee$: the dual group;
$d: \mathcal{N}(G) \to \mathcal{N}(G^\vee)$: order-reversing duality of nilpotent orbits;

Further terminology is explained below. Each page has a graph of the partially ordered set $\mathcal{N}(G)$:

Nodes:
Each node corresponds to a nilpotent orbit $\mathcal{O}$
Dark Blue: $\mathcal{O}$ is even ($\Rightarrow$ special)
Light Blue: $\mathcal{O}$ is special, but not even
white: $\mathcal{O}$ is not special
red border: $d(\mathcal{O})$ is even

Edges:
Edges are closure relations: $\mathcal{O}_2 \in \overline{\mathcal{O}_1}$
green edge: $\mathcal{O}_1, \mathcal{O}_2$ are in the same special piece

Further information and terminology:

$\mathcal{O} \in \mathcal{N}(G)$ is *special* if $\mathcal{O} = d(\mathcal{O}^\vee)$ for some $\mathcal{O}^\vee \in \mathcal{N}(G^\vee)$;
$\mathcal{N}_s(G)$: the special orbits; d is a bijection between $\mathcal{N}_s(G)$ and $\mathcal{N}_s(G^\vee)$;
the *special piece* of $\mathcal{O}$ is the set of orbits $\mathcal{O}'$ satisfying $d(\mathcal{O}) = d(\mathcal{O}')$; this consists of a single special orbit $\mathcal{O}_s$, together with non-special orbits in the closure of $\mathcal{O}_s$, but not in the closure of any smaller special orbit
$A(\mathcal{O}) = \mathrm{Cent}_G(X)/\mathrm{Cent}_G(X)^0$ $(X \in \mathcal{O})$;
$\overline{A}(\mathcal{O})$ (for $\mathcal{O}$ special): Lusztig's canonical quotient of $A(\mathcal{O})$ (cf. [7]), $\overline{A}(\mathcal{O}) \simeq \overline{A}(d(\mathcal{O}))$;
the last line of each node is: $\overline{A}(\mathcal{O})$ (if $\mathcal{O}$ is special), followed by $(A(\mathcal{O}), A(d(\mathcal{O}))$.

D: distinguished (intersects no proper Levi)
R: rigid (not induced)
bi-R: birationally rigid (not birationally induced, i.e. the corresponding moment map is birational, cf. [4]);
q-D: quasi-distinguished ([10]): a unipotent element u is *quasi-distinguished* if there is a semisimple element t so that $tu = ut$ and u is distinguished in the centralizer of t. Quasi-distinguished implies distinguished (take $t = 1$), and implies that the reductive part of the centralizer of u is a torus.

Some implications which are known:

- distinguished $\Rightarrow$ even $\Rightarrow$ special (cf. [6, Theorem 8.2.3])
- even $\Rightarrow$ birationally induced from (the 0-orbit on) a proper Levi ([1, Lemma 27.8])

- quasi-distinguished ⇒ even *except* for two cases in E_8 (case-by-case)

Nota Bene: I am taking the pictures in [12] as being authoritative. These are reproduced in [5]; however there are some mistakes in the latter. Earlier references for E_6, E_7 and E_8 are [8] and [9]. These pictures also have a few errors. The pictures are determined by the Green functions, which are computed in [3]. Also see [2], [6], and [11].

There are two versions of the E_8 pictures; which one you use depends on whether you are less than 50 years old or not.

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References

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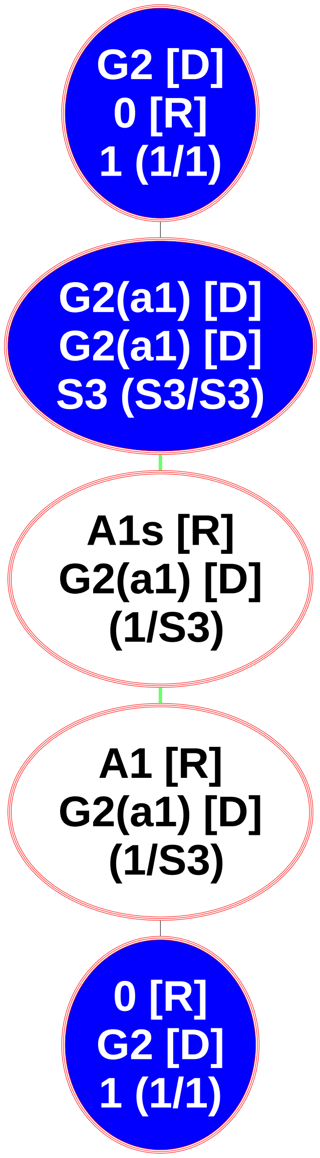
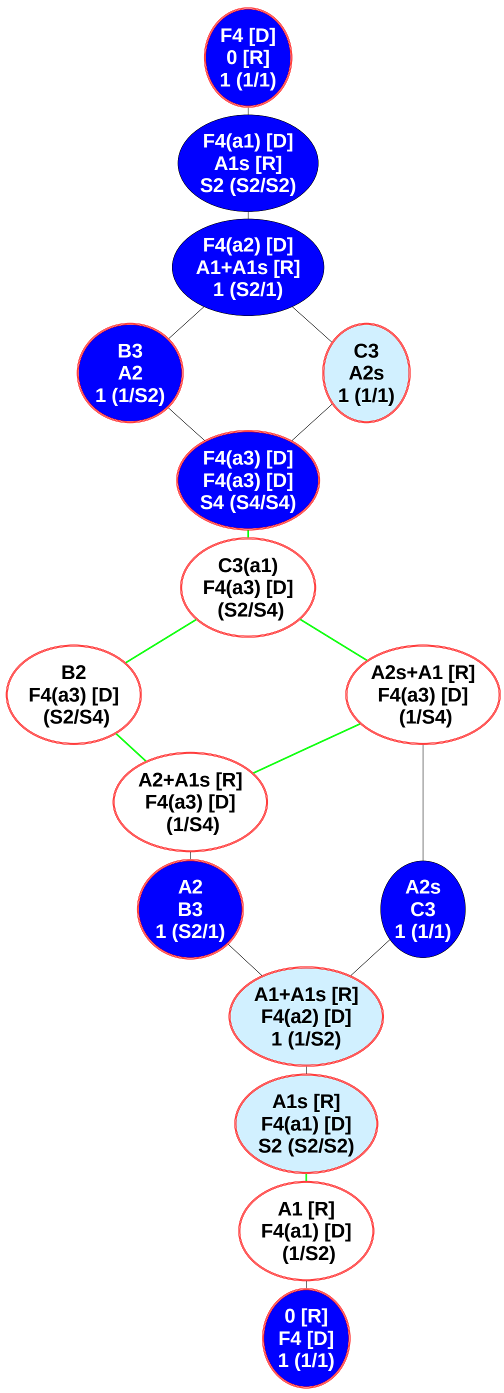
G_2

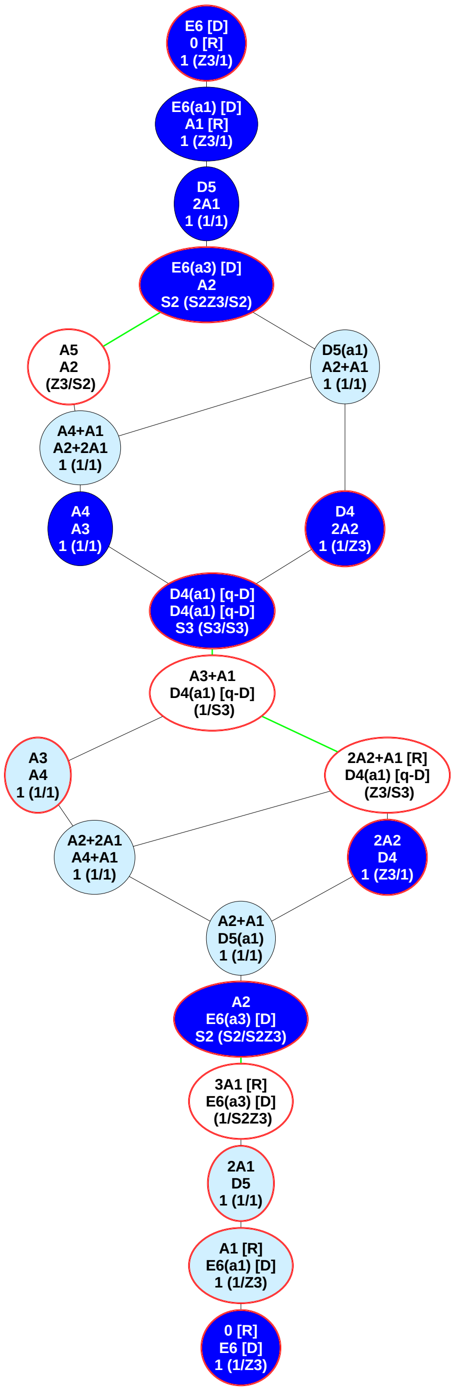

G2 [D]
0 [R]
1 (1/1)
G2(a1) [D]
G2(a1) [D]
S3 (S3/S3)
A1s [R]
G2(a1) [D]
(1/S3)
A1 [R]
G2(a1) [D]
(1/S3)
0 [R]
G2 [D]
1 (1/1)

F_4



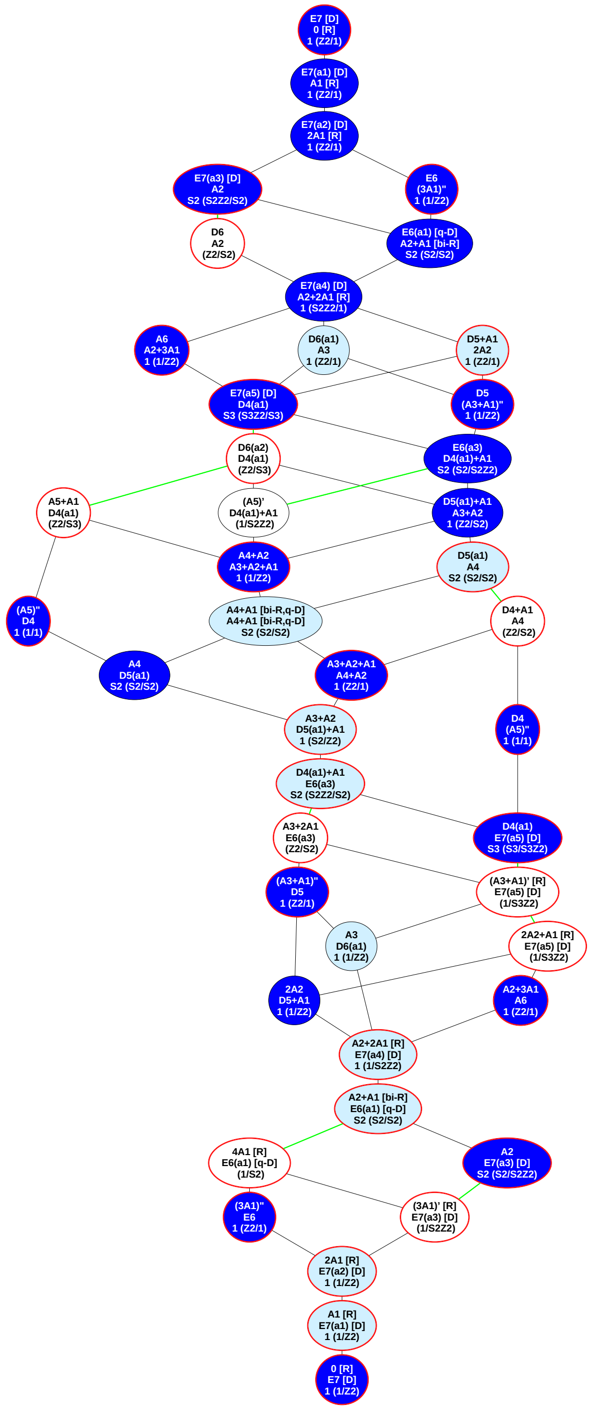
F4 [D]
0 [R]
1 (1/1)
F4(a1) [D]
A1s [R]
S2 (S2/S2)
F4(a2) [D]
A1+A1s [R]
1 (S2/1)
B3
A2
1 (1/S2)
C3
A2s
1 (1/1)
F4(a3) [D]
F4(a3) [D]
S4 (S4/S4)
C3(a1)
F4(a3) [D]
(S2/S4)
B2
F4(a3) [D]
(S2/S4)
A2s+A1 [R]
F4(a3) [D]
(1/S4)
A2+A1s [R]
F4(a3) [D]
(1/S4)
A2
B3
1 (S2/1)
A2s
C3
1 (1/1)
A1+A1s [R]
F4(a2) [D]
1 (1/S2)
A1s [R]
F4(a1) [D]
S2 (S2/S2)
A1 [R]
F4(a1) [D]
(1/S2)
0 [R]
F4 [D]
1 (1/1)

E_6



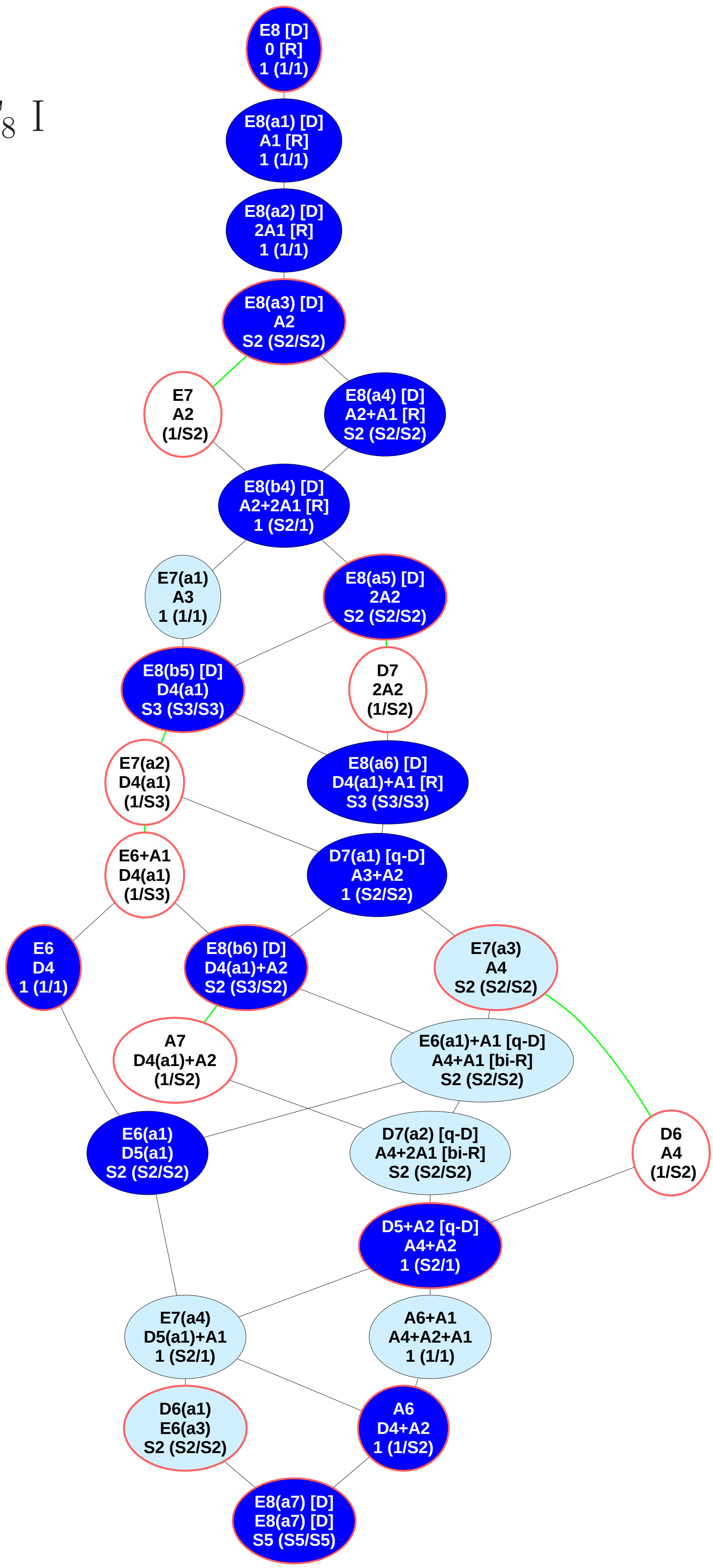
E6 [D]
0 [R]
1 (Z3/1)
E6(a1) [D]
A1 [R]
1 (Z3/1)
D5
2A1
1 (1/1)
E6(a3) [D]
A2
S2 (S2Z3/S2)
A5
A2
(Z3/S2)
D5(a1)
A2+A1
1 (1/1)
A4+A1
A2+2A1
1 (1/1)
A4
A3
1 (1/1)
D4
2A2
1 (1/Z3)
D4(a1) [q-D]
D4(a1) [q-D]
S3 (S3/S3)
A3+A1
D4(a1) [q-D]
(1/S3)
A3
A4
1 (1/1)
2A2+A1 [R]
D4(a1) [q-D]
(Z3/S3)
A2+2A1
A4+A1
1 (1/1)
2A2
D4
1 (Z3/1)
A2+A1
D5(a1)
1 (1/1)
A2
E6(a3) [D]
S2 (S2/S2Z3)
3A1 [R]
E6(a3) [D]
(1/S2Z3)
2A1
D5
1 (1/1)
A1 [R]
E6(a1) [D]
1 (1/Z3)
0 [R]
E6 [D]
1 (1/Z3)

E_7

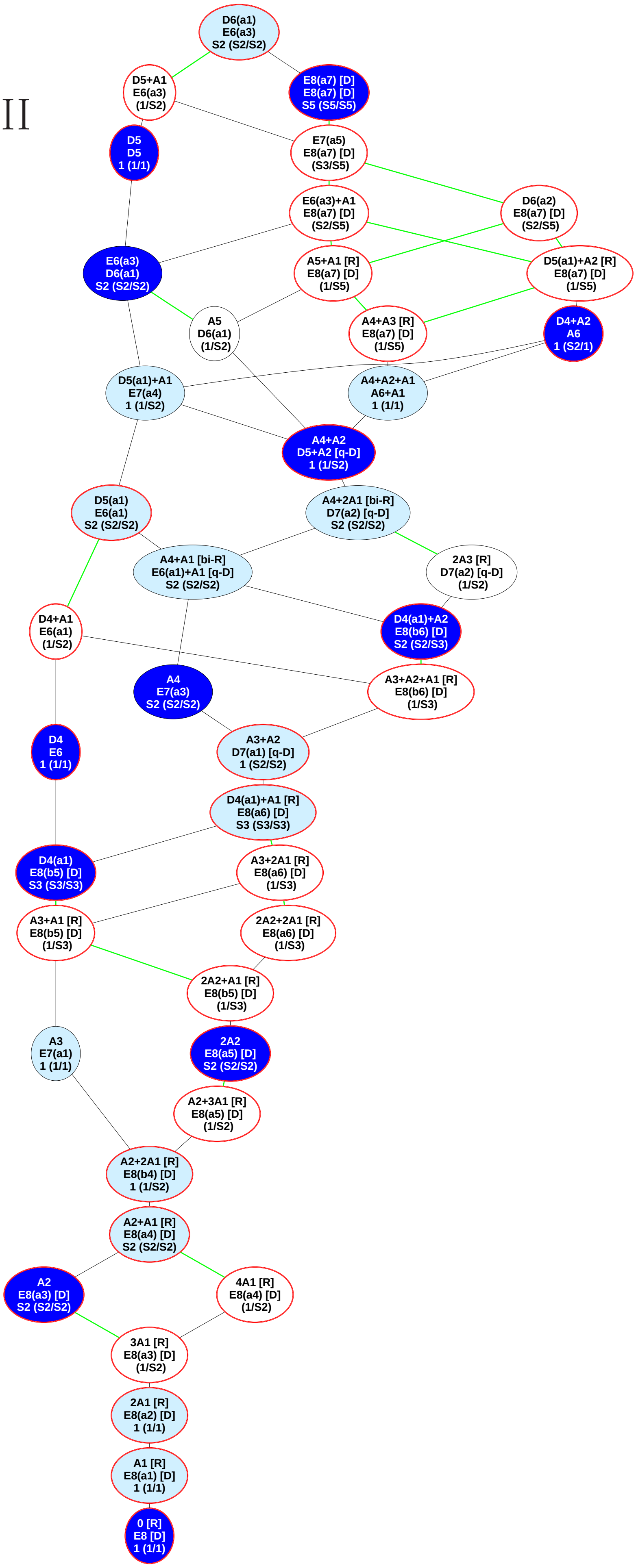


E7 [D] 0 [R] 1 (Z2/1)
E7(a1) [D] A1 [R] 1 (Z2/1)
E7(a2) [D] 2A1 [R] 1 (Z2/1)
E7(a3) [D] A2 S2 (S2Z2/S2)
E6 (3A1)" 1 (1/Z2)
D6 A2 (Z2/S2)
E6(a1) [q-D] A2+A1 [bi-R] S2 (S2/S2)
E7(a4) [D] A2+2A1 [R] 1 (S2Z2/1)
A6 A2+3A1 1 (1/Z2)
D6(a1) A3 1 (Z2/1)
D5+A1 2A2 1 (Z2/1)
E7(a5) [D] D4(a1) S3 (S3Z2/S3)
D5 (A3+A1)" 1 (1/Z2)
D6(a2) D4(a1) (Z2/S3)
E6(a3) D4(a1)+A1 S2 (S2/S2Z2)
A5+A1 D4(a1) (Z2/S3)
(A5)' D4(a1)+A1 (1/S2Z2)
D5(a1)+A1 A3+A2 1 (Z2/S2)
A4+A2 A3+A2+A1 1 (1/Z2)
D5(a1) A4 S2 (S2/S2)
(A5)" D4 1 (1/1)
A4+A1 [bi-R,q-D] A4+A1 [bi-R,q-D] S2 (S2/S2)
D4+A1 A4 (Z2/S2)
A4 D5(a1) S2 (S2/S2)
A3+A2+A1 A4+A2 1 (Z2/1)
A3+A2 D5(a1)+A1 1 (S2/Z2)
D4 (A5)" 1 (1/1)
D4(a1)+A1 E6(a3) S2 (S2Z2/S2)
A3+2A1 E6(a3) (Z2/S2)
D4(a1) E7(a5) [D] S3 (S3/S3Z2)
(A3+A1)" D5 1 (Z2/1)
(A3+A1)' [R] E7(a5) [D] (1/S3Z2)
A3 D6(a1) 1 (1/Z2)
2A2+A1 [R] E7(a5) [D] (1/S3Z2)
2A2 D5+A1 1 (1/Z2)
A2+3A1 A6 1 (Z2/1)
A2+2A1 [R] E7(a4) [D] 1 (1/S2Z2)
A2+A1 [bi-R] E6(a1) [q-D] S2 (S2/S2)
4A1 [R] E6(a1) [q-D] (1/S2)
A2 E7(a3) [D] S2 (S2/S2Z2)
(3A1)" E6 1 (Z2/1)
(3A1)' [R] E7(a3) [D] (1/S2Z2)
2A1 [R] E7(a2) [D] 1 (1/Z2)
A1 [R] E7(a1) [D] 1 (1/Z2)
0 [R] E7 [D] 1 (1/Z2)

E_8 I


E8 [D]
0 [R]
1 (1/1)
E8(a1) [D]
A1 [R]
1 (1/1)
E8(a2) [D]
2A1 [R]
1 (1/1)
E8(a3) [D]
A2
S2 (S2/S2)
E7
A2
(1/S2)
E8(a4) [D]
A2+A1 [R]
S2 (S2/S2)
E8(b4) [D]
A2+2A1 [R]
1 (S2/1)
E7(a1)
A3
1 (1/1)
E8(a5) [D]
2A2
S2 (S2/S2)
E8(b5) [D]
D4(a1)
S3 (S3/S3)
D7
2A2
(1/S2)
E7(a2)
D4(a1)
(1/S3)
E8(a6) [D]
D4(a1)+A1 [R]
S3 (S3/S3)
E6+A1
D4(a1)
(1/S3)
D7(a1) [q-D]
A3+A2
1 (S2/S2)
E6
D4
1 (1/1)
E8(b6) [D]
D4(a1)+A2
S2 (S3/S2)
E7(a3)
A4
S2 (S2/S2)
A7
D4(a1)+A2
(1/S2)
E6(a1)+A1 [q-D]
A4+A1 [bi-R]
S2 (S2/S2)
E6(a1)
D5(a1)
S2 (S2/S2)
D7(a2) [q-D]
A4+2A1 [bi-R]
S2 (S2/S2)
D6
A4
(1/S2)
D5+A2 [q-D]
A4+A2
1 (S2/1)
E7(a4)
D5(a1)+A1
1 (S2/1)
A6+A1
A4+A2+A1
1 (1/1)
D6(a1)
E6(a3)
S2 (S2/S2)
A6
D4+A2
1 (1/S2)
E8(a7) [D]
E8(a7) [D]
S5 (S5/S5)

E_8 II



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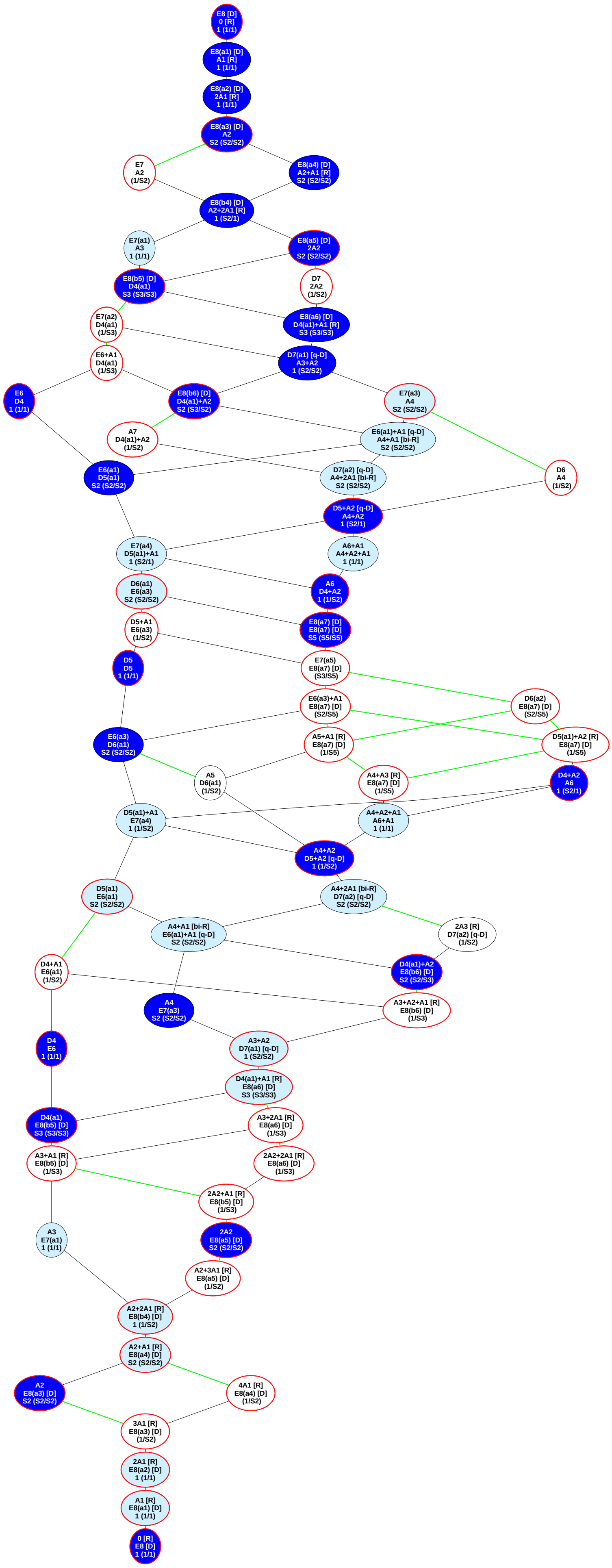
D5+A1 E6(a3) (1/S2)
E8(a7) [D] E8(a7) [D] S5 (S5/S5)
D5 D5 1 (1/1)
E7(a5) E8(a7) [D] (S3/S5)
E6(a3)+A1 E8(a7) [D] (S2/S5)
D6(a2) E8(a7) [D] (S2/S5)
E6(a3) D6(a1) S2 (S2/S2)
A5+A1 [R] E8(a7) [D] (1/S5)
D5(a1)+A2 [R] E8(a7) [D] (1/S5)
A5 D6(a1) (1/S2)
A4+A3 [R] E8(a7) [D] (1/S5)
D4+A2 A6 1 (S2/1)
D5(a1)+A1 E7(a4) 1 (1/S2)
A4+A2+A1 A6+A1 1 (1/1)
A4+A2 D5+A2 [q-D] 1 (1/S2)
D5(a1) E6(a1) S2 (S2/S2)
A4+2A1 [bi-R] D7(a2) [q-D] S2 (S2/S2)
A4+A1 [bi-R] E6(a1)+A1 [q-D] S2 (S2/S2)
2A3 [R] D7(a2) [q-D] (1/S2)
D4+A1 E6(a1) (1/S2)
D4(a1)+A2 E8(b6) [D] S2 (S2/S3)
A4 E7(a3) S2 (S2/S2)
A3+A2+A1 [R] E8(b6) [D] (1/S3)
D4 E6 1 (1/1)
A3+A2 D7(a1) [q-D] 1 (S2/S2)
D4(a1)+A1 [R] E8(a6) [D] S3 (S3/S3)
D4(a1) E8(b5) [D] S3 (S3/S3)
A3+2A1 [R] E8(a6) [D] (1/S3)
A3+A1 [R] E8(b5) [D] (1/S3)
2A2+2A1 [R] E8(a6) [D] (1/S3)
2A2+A1 [R] E8(b5) [D] (1/S3)
A3 E7(a1) 1 (1/1)
2A2 E8(a5) [D] S2 (S2/S2)
A2+3A1 [R] E8(a5) [D] (1/S2)
A2+2A1 [R] E8(b4) [D] 1 (1/S2)
A2+A1 [R] E8(a4) [D] S2 (S2/S2)
A2 E8(a3) [D] S2 (S2/S2)
4A1 [R] E8(a4) [D] (1/S2)
3A1 [R] E8(a3) [D] (1/S2)
2A1 [R] E8(a2) [D] 1 (1/1)
A1 [R] E8(a1) [D] 1 (1/1)
0 [R] E8 [D] 1 (1/1)

E_8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E8(a2) [D] 2A1 [R] 1 (1/1)
E8(a3) [D] A2 S2 (S2/S2)
E7 A2 (1/S2)
E8(a4) [D] A2+A1 [R] S2 (S2/S2)
E8(b4) [D] A2+2A1 [R] 1 (S2/1)
E7(a1) A3 1 (1/1)
E8(a5) [D] 2A2 S2 (S2/S2)
E8(b5) [D] D4(a1) S3 (S3/S3)
D7 2A2 (1/S2)
E7(a2) D4(a1) (1/S3)
E8(a6) [D] D4(a1)+A1 [R] S3 (S3/S3)
E6+A1 D4(a1) (1/S3)
D7(a1) [q-D] A3+A2 1 (S2/S2)
E6 D4 1 (1/1)
E8(b6) [D] D4(a1)+A2 S2 (S3/S2)
E7(a3) A4 S2 (S2/S2)
A7 D4(a1)+A2 (1/S2)
E6(a1)+A1 [q-D] A4+A1 [bi-R] S2 (S2/S2)
E6(a1) D5(a1) S2 (S2/S2)
D7(a2) [q-D] A4+2A1 [bi-R] S2 (S2/S2)
D6 A4 (1/S2)
D5+A2 [q-D] A4+A2 1 (S2/1)
E7(a4) D5(a1)+A1 1 (S2/1)
A6+A1 A4+A2+A1 1 (1/1)
D6(a1) E6(a3) S2 (S2/S2)
A6 D4+A2 1 (1/S2)
D5+A1 E6(a3) (1/S2)
E8(a7) [D] E8(a7) [D] S5 (S5/S5)
D5 D5 1 (1/1)
E7(a5) E8(a7) [D] (S3/S5)
E6(a3)+A1 E8(a7) [D] (S2/S5)
D6(a2) E8(a7) [D] (S2/S5)
E6(a3) D6(a1) S2 (S2/S2)
A5+A1 [R] E8(a7) [D] (1/S5)
D5(a1)+A2 [R] E8(a7) [D] (1/S5)
A5 D6(a1) (1/S2)
A4+A3 [R] E8(a7) [D] (1/S5)
D4+A2 A6 1 (S2/1)
D5(a1)+A1 E7(a4) 1 (1/S2)
A4+A2+A1 A6+A1 1 (1/1)
A4+A2 D5+A2 [q-D] 1 (1/S2)
D5(a1) E6(a1) S2 (S2/S2)
A4+2A1 [bi-R] D7(a2) [q-D] S2 (S2/S2)
A4+A1 [bi-R] E6(a1)+A1 [q-D] S2 (S2/S2)
2A3 [R] D7(a2) [q-D] (1/S2)
D4+A1 E6(a1) (1/S2)
D4(a1)+A2 E8(b6) [D] S2 (S2/S3)
A4 E7(a3) S2 (S2/S2)
A3+A2+A1 [R] E8(b6) [D] (1/S3)
D4 E6 1 (1/1)
A3+A2 D7(a1) [q-D] 1 (S2/S2)
D4(a1)+A1 [R] E8(a6) [D] S3 (S3/S3)
D4(a1) E8(b5) [D] S3 (S3/S3)
A3+2A1 [R] E8(a6) [D] (1/S3)
A3+A1 [R] E8(b5) [D] (1/S3)
2A2+2A1 [R] E8(a6) [D] (1/S3)
2A2+A1 [R] E8(b5) [D] (1/S3)
A3 E7(a1) 1 (1/1)
2A2 E8(a5) [D] S2 (S2/S2)
A2+3A1 [R] E8(a5) [D] (1/S2)
A2+2A1 [R] E8(b4) [D] 1 (1/S2)
A2+A1 [R] E8(a4) [D] S2 (S2/S2)
A2 E8(a3) [D] S2 (S2/S2)
4A1 [R] E8(a4) [D] (1/S2)
3A1 [R] E8(a3) [D] (1/S2)
2A1 [R] E8(a2) [D] 1 (1/1)
A1 [R] E8(a1) [D] 1 (1/1)
0 [R] E8 [D] 1 (1/1)