Cells, Orbits, Primitive Ideals and W-Representations

Let G be a real form of a linear connected complex algebraic group $G_{\mathbb{C}}$ defined over $\mathbb{R}$, and let $\widehat{G}_{adm,\lambda}$ be the set of (equivalence classes of) irreducible admissible representation of G of infinitesimal character λ, which we assume to be regular and integral. One can endow $\widehat{G}_{adm,\lambda}$ with the structure of a W-graph $\mathcal{G}_\lambda$ as follows:

- the **vertices** of $\mathcal{G}$ correspond to irreducible admissible representations in $\widehat{G}_{adm,\lambda}$
- Two vertices $x, y \in \widehat{G}_{adm,\lambda}$ are linked by a (directed) **edge** $x \to y$ if x occurs as a subquotient of $y \otimes \mathfrak{g}$.
- The **multiplicity** of an edge $x \to y$ is equal to the multiplicity with which x occurs as a subquotient of $y \otimes \mathfrak{g}$
- The τ**-invariant** of a vertex x is equal to the τ-invariant of $Ann_{U(\mathfrak{g})}(x)$ (that is, the tau invariant of the annihilator in $U(\mathfrak{g})$ of the irreducible $(\mathfrak{g}, K)$-module corresponding to $x \in \widehat{G}_{adm,\lambda}$, a subset of the simple roots of $\mathfrak{g}$).

The connected components of the W-graph $\mathcal{G}_\lambda$ corresponded to certain subsets of $\widehat{G}_{adm,\lambda}$ known as **blocks** of (irreducible admissible) representations. Blocks in $\widehat{G}_{adm,\lambda}$ are effectively parameterized by the set of strong real forms of the complex algebraic group $(G_{\mathbb{C}})^\vee$ dual to $G_{\mathbb{C}}$ corresponding to λ (albeit, some of these blocks may be empty).

The strongly connected components of the W-graph $\mathcal{G}_\lambda$, that is, the maximal sets of vertices that can be connected bidirectionally by sequences of directed edges are known as **cells**.

The Atlas software computes the W-graph of $\widehat{G}_{adm,\lambda}$ block by block. The tables given below relate the output of such computations to nilpotent orbits, primitive ideals and Weyl group representations.

1. Computational Scope

In the tables given below are meant to be complete for the real reductive Lie groups G whose complexified Lie algebras coincide with one of the exceptional Lie algebras.

The first thing I did was to run the Atlas `blocksizes` command to identify all of the nontrivial blocks of representations for the each real forms of $G_{\mathbb{C}}$[1]. In the results appearing below, the real forms of $G_{\mathbb{C}}$ are listed in the left most column and corresponding dual real forms are listed in the first row. The integer entries in the interior correspond to the number of representations in the corresponding block.

- E_6 : (simply connected), split inner class.

	$\mathfrak{e}_6$	$E_6(\mathfrak{f}_4)$	$E_6(\mathbb{R})$
$E_6(F_4)$	0	0	45
$E_6(\mathbb{R})$	1	513	1881

- E_6 : (simply connected), compact inner class

	$E_6(F_4)$	$E_6(\mathbb{R})$
$\mathfrak{e}_6$	0	1
$E_6(\mathfrak{f}_4)$	0	513
$E_6(\mathbb{R})$	45	1881

[1]Parentheses are used to indicate the insignificant choices that were made in setting up the inner classes.

- E_7 : simply connected, (split inner class)

	$\mathfrak{e}_7$	$E_7(\mathfrak{e}_6.\mathfrak{u}(1))$	$E_7(\mathfrak{so}(12).\mathfrak{su}(2))$	$E_7(\mathbb{R})$
$\mathfrak{e}_7$	0	0	0	1
$E_7(\mathfrak{e}_6.\mathfrak{u}(1))$	0	0	315	3017
$E_7(\mathfrak{so}(12).\mathfrak{su}(2))$	0	315	0	9576
$E_7(\mathbb{R})$	1	1981	9576	24678

- E_7 : adjoint, (compact inner class)

	$\mathfrak{e}_7$	$E_7(\mathfrak{e}_6.\mathfrak{u}(1))$	$E_7(\mathfrak{so}(12).\mathfrak{su}(2))$	$E_7(\mathbb{R})$
$\mathfrak{e}_7$	0	0	0	1
$E_7(\mathfrak{e}_6.\mathfrak{u}(1))$	0	0	315	1981
$E_7(\mathfrak{so}(12).\mathfrak{su}(2))$	0	315	0	9576
$E_7(\mathbb{R})$	1	3017	9576	24678

- E_8 : (simply connected), (split inner class)

	$\mathfrak{e}_8$	$E_8(\mathfrak{e}_7.\mathfrak{su}(2))$	$E_8(\mathbb{R})$
$\mathfrak{e}_8$	0	0	1
$E_8(\mathfrak{e}_7.\mathfrak{su}(2))$	0	3150	74410
$E_8(\mathbb{R})$	1	73410	453060

- F_4 : (simply connected), (split inner class)

	$\mathfrak{f}_4$	$F_4(\mathfrak{so}(9))$	$F_4(\mathbb{R})$
$\mathfrak{f}_4$	0	0	1
$F_4(\mathfrak{so}(9))$	0	0	15
$F_4(\mathbb{R})$	1	15	336

- G_2 : (simply connected), (split inner class)

	$\mathfrak{g}_2$	$G_2(\mathbb{R})$
$\mathfrak{g}_2$	0	1
$G_2(\mathbb{R})$	1	12

On each of the blocks listed above, I then ran the Atlas `wcells` command. This command decomposes the vertices of a block into cells and then ascribes to each cell vertex its τ-invariant, edges, and edge-multiplicities.

1.1. **Cell Invariants.**

With the `wcell` data in hand, I next proceeded to compute the following invariants, using Maple and John Stembridge's Coxeter/Weyl packages.

1.1.1. *Cardinality.*

Definition: The number of representations in a cell.

Algorithm: count

1.1.2. *Attached Nilpotent Orbit.*

Definition: The primitive ideals in $U(\mathfrak{g})$ corresponding to the annihilators of the representations in a cell C have the share the same associated variety and that associated variety is the closure of a single nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}^*$. This orbit is the nilpotent orbit attached to a cell.

Algorithm: In the case of regular infinitesimal characters, the attached orbits are always special nilpotent orbits. These in turn can be uniquely specified by prescribing the minimal Richardson orbits whose closures contain them and the Richardson orbits whose closures contain their Spaltenstein dual. This prescription in turn amounts to a specification of certain subsets of simple roots: for a Richardson orbit is an orbit of the form $ind^{\mathfrak{g}}_{\mathfrak{l}_\Gamma}(\mathbf{0})$, where Γ is a subset of the simple roots Π of $\mathfrak{g}$. The **tau signature** of a special orbit $\mathcal{O}$ is defined as the pair $(\tau(\mathcal{O}), \tau(\mathcal{O}^\vee))$ where

$$\tau(\mathcal{O}) = \left\{\Gamma \subset \Pi \mid \mathcal{O} \subset \overline{ind^{\mathfrak{g}}_{\mathfrak{l}_\Gamma}(\mathbf{0})}\right\}$$

A similar τ-signature $(\tau(C), \tau^\vee(C))$ was defined for cells,

$$\begin{aligned}\tau(C) &= \{\Gamma \subset \Pi \mid \Gamma \text{ occurs in the set of } \tau\text{-invariants of } C\}\\ \tau^\vee(C) &= \{\Gamma \subset \Pi \mid \Gamma \text{ occurs in the set of complements of the } \tau\text{-invariants in } C\}\end{aligned}$$

By results of Spaltenstein and David, a necessary and sufficient for a special nilpotent orbit $\mathcal{O}$ to be attached to a cell C is

$$(\tau(\mathcal{O}), \tau(\mathcal{O}^\vee)) = (\tau(C), \tau^\vee(C))\, O$$

Orbits were attached to cells by explicitly computing the tau signatures of orbits and cells and then matching them up.

1.1.3. *Special Representations.*

Definition: The Springer correspondence attaches a unique representation of the Weyl group to each nilpotent orbit. The representations attached to special nilpotent orbits turn out to be special representations of the Weyl group. (I should point out that the notion of a special representation of W is independent of the Springer correspondence.)

Algorithm: Once nilpotent orbits were attached to cells, the tables in [Carter, *Finite Groups of Lie Type*] were used to assign special representations to cells.

1.1.4. *τ_∞-partitioning.*

Definition: Let $\tau_0(x)$ be the τ-invariant of a vertex x of a cell C, and let $\varepsilon(x)$ be the set of vertices y for which $x \to y$ is an edge. Set

$$\begin{aligned}\tau_1(x) &= \{\tau_0(y) \mid y \in \varepsilon(x)\}\\ \tau_2(x) &= \{\tau_1(y) \mid y \in \varepsilon(x)\}\\ &\vdots\\ \tau_k(x) &= \{\tau_{k-1}(y) \mid y \in \varepsilon(x)\}\end{aligned}$$

Let P_k be the partitioning of a cell corresponding to the equivalence relation

$$x \sim_K x' \quad \Longleftrightarrow \quad \tau_i(x) = \tau_i(x') \quad \text{for } i = 0, 1, \ldots, k$$

Then P_k is a refinement of P_{k-1}. Since the cells are finite in cardinality, eventually $P_k = P_{k+1} = \cdots$ for all k sufficiently large. We call the resulting stable partitioning of C, the τ_∞ partitioning of C. It is compatible with the partitioning of a cell into subsets of representations sharing the same primitive ideal (i.e. the partitioning of C by the equivalence relation $x \sim_{PI} x' \quad \Longleftrightarrow \quad Ann_{U(\mathfrak{g})}(x) = Ann_{U(\mathfrak{g})}(x')$).

Algorithm: straight-forward application of the definition to W-graph of a cell (from `wcells` data). The "answer" is most succinctly expressed as a partition of $|C|$; e.g. I use

$$\tau_\infty(C) = 1^{84}2^{336}$$

to indicate a 756-element cell breaks up into 84 singlets and 336 doublets under the τ_∞-partitioning scheme.

Observation: In all cases the number of τ_∞-equivalence classes of in a cell coincides with the maximal number of distinct primitive ideals (= dimension of the special Weyl group representation attached to the nilpotent orbit attached to the cell) that can occur. Since the partitioning of the cell by primitive ideals is at worst a refinement of the τ_∞-partitioning, the fact that

$$\#\left\{\tau_\infty\text{-equivalence classes}\right\} = \text{maximal number of distinct primitive ideals that can occur in } C$$

implies that the τ_∞ partitioning and the partitioning by primitive ideals coincide.

1.1.5. *Cell Representation.*

Definition: The W-graph of a cell prescribes a representation of the Weyl group W of $\mathfrak{g}$, induced by the coherent continuation representation of W on the parameter set $\mathcal{HC}_\lambda$.

Algorithm: Let Γ_C be the free $\mathbb{Z}$-module freely generated by the irreducible x in C. For each $i \in \Pi$, define an action of the simple Weyl reflections on Γ_C by

$$T_i(x) = \begin{cases} -x & \text{if } i \in \tau(x) \\ x + \sum_{\substack{y \in \varepsilon(x) \\ i \notin \tau(y)}} m_{y \to x} y & \text{if } i \notin \tau(x) \end{cases}$$

This action on generators satisfies both quadratic relations and braid relations of W and so yeilds a representation of W. By explicitly constructing a set of $|C| \times |C|$ matrices implementing this action and then computing the traces of the representatives of each conjugacy class in W, the character of the cell representation can be computed. Expressing this character as a sum of irreducible characters yields the cell representation.

Observation: In writing down the cell representations, I utilized a parameterization of irreducible Weyl group representations by pairs (D, d) where D is the dimension of the representation and d is the degree at which the representation first occurs in $S(\mathfrak{h})$. (With just a couple of exceptions this data is sufficient to separate the representations of W.) This turned out to be a fortuitous choice for

- If σ_s is the special representation that occurs in a cell representation (each cell representation has one and only special representation as a constituent), then

$$\text{degree } d \text{ in which } \sigma_s \text{ first occurs in } S(\mathfrak{h}) = \left|\Delta^+\right| - \frac{1}{2}\dim\left(\mathcal{O}_C\right)$$

- If σ' is any other irreducible representation occuring in a cell representation then

$$\text{degree } d \text{ in which } \sigma' \text{ first occurs in } S(\mathfrak{h}) = \left|\Delta^+\right| - \frac{1}{2}\dim\left(\mathcal{O}_C\right) + \varepsilon$$

  by which I mean σ' first occurs in $S(\mathfrak{h})$ at a degree that is only slightly higher than that of the special representation. In what follows, I shall make some wild speculations about what this might mean.

2. Tables

Below we give a series of tables for the various blocks of irreducible Harish-Chandra modules of the exceptional groups. Here is how the data is organized. Each row in each table corresponds to a particular cell in a particular block of an exceptional group. In cases where the adjoint group and the simply connected groups produce the same table we present only the table for the simply connected group. Similarly, whenever the same table results for the split and compact inner classes, we present only the table for the split inner class. The data in the columns of each table is as follows:

\# the cell number as enumerated in the output of the atlas `extract-cells` command

$|C|$ the cardinality of the cell (i.e. the number of irreducible representations in the cell)

$\mathcal{O}_C$ the special orbit attached to the cell - computed by determining the minimal Richardson orbits containing the cell's orbit and the minimal Richardson orbits containing the orbit of the corresponding dual cell)

n the order in which the orbit $\mathcal{O}_C$ appears in the Collingwood and McGoverns tables of nilpotent orbits (pgs. 128 – 134 of [CW]).

σ_{spec} the special representation attached to the cell - determined by applying the Springer correspondence to $\mathcal{O}_C$

$\#Prim\,(C)$ the number of primitive ideals in the cell (the number of distinct subcells in the τ_∞ partitioning of the cell)

$\mathcal{P}_{\tau_\infty}$ the τ_∞ partitioning of the cell written as an actual partition of $|C|$. E.g. $1^{20}2^{50}3^{10}$ means the τ_∞ partitioning of the cell decomposes it into 20 singlets, 50 doublets, and 10 triplets of representations sharing the same primitive ideal.

W-rep the representation of the Weyl group carried by the cell. This is the coherent continuation representation inferred from the W-graph structure of the cell. It was computed by using the output of `extract-cells` to explicitly construct the (restriction of the) coherent continuation representation to the cell, computing the character of this representation by taking the traces of representatives of each conjugacy class in W, and then writing this character into a sum irreducible characters.

The last series of table contains the analogous results for the complex exceptional groups, excluding $E_8\,(\mathbb{C})$.

2.1. G_2.

11 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	G_2	5	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	5	$G_2(a_1)$	4	$\phi_{2,1}$	2	$2^1 3^1$	$\phi_{2,1}+\phi_{2,2}+\phi'_{1,3}$
2	5	$G_2(a_1)$	4	$\phi_{2,1}$	2	$2^1 3^1$	$\phi_{2,1}+\phi_{2,2}+\phi''_{1,3}$
3	1	0	1	$\phi_{1,6}$	1	1^1	$\phi_{1,6}$

2.2. F_4.

12 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	8	$\tilde{A}_2$	6	$\phi'_{8,9}$	8	1^8	$\phi'_{8,9}$
1	6	$\tilde{A}_1$	3	$\phi_{4,13}$	4	$1^2 2^2$	$\phi_{4,13}+\phi'_{2,16}$
2	1	0	1	$\phi_{1,24}$	1	1^1	$\phi_{1,24}$

21 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	F_4	16	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	6	$F_4(a_1)$	15	$\phi_{4,1}$	4	$1^2 2^2$	$\phi_{4,1}+\phi'_{2,4}$
2	8	C_3	13	$\phi'_{8,3}$	8	1^8	$\phi'_{8,3}$

22 Block

#	$\|C\|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	F_4	16	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	6	$F_4(a_1)$	15	$\phi_{4,1}$	4	$1^2 2^2$	$\phi_{4,1}+\phi''_{2,4}$
3	6	$F_4(a_1)$	15	$\phi_{4,1}$	4	$1^2 2^2$	$\phi_{4,1}+\phi''_{2,4}$
4	9	$F_4(a_2)$	14	$\phi_{9,2}$	9	1^9	$\phi_{9,2}$
5	9	$F_4(a_2)$	14	$\phi_{9,2}$	9	1^9	$\phi_{9,2}$
11	9	$F_4(a_2)$	14	$\phi_{9,2}$	9	1^9	$\phi_{9,2}$
12	8	C_3	13	$\phi'_{8,3}$	8	1^8	$\phi'_{8,3}$
2	8	B_3	12	$\phi''_{8,3}$	8	1^8	$\phi''_{8,3}$
6	8	B_3	12	$\phi''_{8,3}$	8	1^8	$\phi''_{8,3}$
7	8	B_3	12	$\phi''_{8,3}$	8	1^8	$\phi''_{8,3}$
8	8	B_3	12	$\phi''_{8,3}$	8	1^8	$\phi''_{8,3}$
9	57	$F_4(a_3)$	11	$\phi_{12,4}$	12	$2^3 3^1 5^3 6^4 9^1$	$\phi_{12,4}+\phi_{16,5}+2\phi''_{9,6}+\phi''_{6,6}+\phi''_{4,7}+\phi''_{1,12}$
13	47	$F_4(a_3)$	11	$\phi_{12,4}$	12	$5^4 4^4 3^3 2^1$	$\phi_{12,4}+\phi_{16,5}+\phi''_{9,6}+\phi'_{6,6}+\phi''_{4,7}$
14	72	$F_4(a_3)$	11	$\phi_{12,4}$	12	$4^6 6^2 9^4$	$\phi_{12,4}+2\phi_{16,5}+\phi'_{9,6}+\phi''_{9,6}+\phi''_{6,6}+\phi_{4,8}$
18	8	$\tilde{A}_2$	6	$\phi'_{8,9}$	8	1^8	$\phi'_{8,9}$
10	8	A_2	5	$\phi''_{8,9}$	8	1^8	$\phi''_{8,9}$
15	8	A_2	5	$\phi''_{8,9}$	8	1^8	$\phi''_{8,9}$
16	8	A_2	5	$\phi''_{8,9}$	8	1^8	$\phi''_{8,9}$
21	8	A_2	5	$\phi''_{8,9}$	8	1^8	$\phi''_{8,9}$
17	9	$A_1+\tilde{A}_1$	4	$\phi_{9,10}$	9	1^9	$\phi_{9,10}$
19	9	$A_1+\tilde{A}_1$	4	$\phi_{9,10}$	9	1^9	$\phi_{9,10}$
20	9	$A_1+\tilde{A}_1$	4	$\phi_{9,10}$	9	1^9	$\phi_{9,10}$
22	6	$\tilde{A}_1$	3	$\phi_{4,13}$	4	$1^2 2^2$	$\phi_{4,13}+\phi''_{2,16}$
23	6	$\tilde{A}_1$	3	$\phi_{4,13}$	4	$1^2 2^2$	$\phi_{4,13}+\phi''_{2,16}$
24	1	0	1	$\phi_{1,24}$	1	1^1	$\phi_{1,24}$

2.3. E_6 : **compact inner class.**

11 Block

#	$\|C\|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
2	81	A_3+A_1	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
3	81	A_3+A_1	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
1	24	$2A_2$	7	$\phi_{24,12}$	24	1^{24}	$\phi_{24,12}$
4	64	A_2+A_1	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
6	64	A_2+A_1	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
8	45	A_2	5	$\phi_{30,15}$	30	$1^{15}2^{15}$	$\phi_{30,15}+\phi_{15,17}$
5	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
7	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
9	20	$2A_1$	3	$\phi_{20,10}$	20	1^{20}	$\phi_{20,20}$
10	6	A_1	2	$\phi_{6,25}$	6	1^6	$\phi_{6,25}$
11	6	A_1	2	$\phi_{6,25}$	6	1^6	$\phi_{6,25}$
12	1	0	1	$\phi_{1,36}$	1	1^1	$\phi_{1,36}$

20 Block

#	$\|C\|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_6	21	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
2	24	D_4	14	$\phi_{24,6}$	24	1^{24}	$\phi_{24,6}$

21 Block

#	$\|C\|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_6	21	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
3	6	$E_6(a_1)$	20	$\phi_{6,1}$	6	1^6	$\phi_{6,1}$
1	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
4	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
5	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3}+\phi_{15,4}$
11	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3}+\phi_{15,4}$
6	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
7	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
12	60	A_4+A_1	15	$\phi_{60,5}$	60	1^{60}	$\phi_{60,5}$
13	60	A_4+A_1	15	$\phi_{60,5}$	60	1^{60}	$\phi_{60,5}$
2	24	D_4	14	$\phi_{24,6}$	24	1^{24}	$\phi_{24,6}$
9	24	D_4	14	$\phi_{24,6}$	24	1^{24}	$\phi_{24,6}$
8	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
14	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
10	230	$D_4(a_1)$	12	$\phi_{80,7}$	80	$2^{10}3^{70}$	$\phi_{80,7}+\phi_{90,8}+\phi_{60,8}$
15	150	$D_4(a_1)$	12	$\phi_{80,7}$	80	$1^{20}2^{50}3^{10}$	$\phi_{80,7}+\phi_{60,8}+\phi_{10,9}$
16	230	$D_4(a_1)$	12	$\phi_{80,7}$	80	$2^{10}3^{70}$	$\phi_{80,7}+\phi_{90,8}+\phi_{60,8}$
19	81	A_3+A_1	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
21	81	A_3+A_1	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
17	60	A_2+2A_1	8	$\phi_{60,11}$	60	1^{60}	$\phi_{60,11}$
18	60	A_2+2A_1	8	$\phi_{60,11}$	60	1^{60}	$\phi_{60,11}$
22	60	A_2+2A_1	8	$\phi_{60,11}$	60	1^{60}	$\phi_{60,11}$
23	24	$2A_2$	7	$\phi_{24,12}$	24	1^{24}	$\phi_{24,12}$
24	64	A_2+A_1	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
25	64	A_2+A_1	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
20	45	A_2	5	$\phi_{30,15}$	30	$1^{15}2^{15}$	$\phi_{30,15}+\phi_{15,16}$
26	45	A_2	5	$\phi_{30,15}$	30	$1^{15}2^{15}$	$\phi_{30,15}+\phi_{15,16}$
27	45	A_2	5	$\phi_{30,15}$	30	$1^{15}2^{15}$	$\phi_{30,15}+\phi_{15,16}$
28	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
29	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
30	6	A_1	2	$\phi_{6,25}$	6	1^6	$\phi_{6,25}$
31	1	0	1	$\phi_{1,36}$	1	1^1	$\phi_{1,36}$

2.4. E_6 : **split inner class.**

02 Block

#	$\|C\|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	24	$2A_2$	7	$\phi_{24,12}$	24	1^{24}	$\phi_{24,12}$
1	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
2	1	0	1	$\phi_{1,36}$	1	1^1	$\phi_{1,36}$

11 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_6	21	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	6	$E_6(a_1)$	20	$\phi_{6,1}$	6	1^6	$\phi_{6,1}$
2	6	$E_6(a_1)$	20	$\phi_{6,1}$	6	1^6	$\phi_{6,1}$
3	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
5	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
7	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
4	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3}+\phi_{15,5}$
6	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
8	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
9	24	D_4	14	$\phi_{24,6}$	24	1^{24}	$\phi_{24,6}$
10	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
11	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
12	81	A_3	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,6}$

12 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_6	21	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	6	$E_6(a_1)$	20	$\phi_{6,1}$	6	1^6	$\phi_{6,1}$
2	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
3	20	D_5	19	$\phi_{20,2}$	20	1^{20}	$\phi_{20,2}$
4	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3}+\phi_{15,4}$
5	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3}+\phi_{15,4}$
15	45	$E_6(a_3)$	18	$\phi_{30,3}$	30	$1^{15}2^{15}$	$\phi_{30,3}+\phi_{15,4}$
6	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
7	64	$D_5(a_1)$	17	$\phi_{64,4}$	64	1^{64}	$\phi_{64,4}$
9	60	A_4+A_1	15	$\phi_{60,5}$	60	1^{60}	$\phi_{60,5}$
10	60	A_4+A_1	15	$\phi_{60,5}$	60	1^{60}	$\phi_{60,5}$
11	60	A_4+A_1	15	$\phi_{60,5}$	60	1^{60}	$\phi_{60,5}$
8	24	D_4	14	$\phi_{24,6}$	24	1^{24}	$\phi_{24,6}$
12	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
16	81	A_4	13	$\phi_{81,6}$	81	1^{81}	$\phi_{81,6}$
13	230	$D_4(a_1)$	12	$\phi_{80,7}$	80	$2^{10}3^{70}$	$\phi_{80,7}+\phi_{90,8}+\phi_{60,8}$
14	150	$D_4(a_1)$	12	$\phi_{80,7}$	80	$1^{20}2^{50}3^{10}$	$\phi_{80,7}+\phi_{60,8}+\phi_{10,9}$
17	230	$D_4(a_1)$	12	$\phi_{80,7}$	80	$2^{10}3^{70}$	$\phi_{80,7}+\phi_{90,8}+\phi_{60,8}$
18	81	A_3+A_1	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
23	81	A_3+A_1	9	$\phi_{81,10}$	81	1^{81}	$\phi_{81,10}$
19	60	A_2+2A_1	8	$\phi_{60,11}$	60	1^{60}	$\phi_{60,11}$
20	60	A_2+2A_1	8	$\phi_{60,11}$	60	1^{60}	$\phi_{60,11}$
22	24	$2A_2$	7	$\phi_{24,12}$	24	1^{24}	$\phi_{24,12}$
27	24	$2A_2$	7	$\phi_{24,12}$	24	1^{24}	$\phi_{24,12}$
24	64	A_2+A_1	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
25	64	A_2+A_1	6	$\phi_{64,13}$	64	1^{64}	$\phi_{64,13}$
21	45	A_2	5	$\phi_{30,15}$	60	$1^{15}2^{15}$	$\phi_{30,15.}+\phi_{15,16}$
26	45	A_2	5	$\phi_{30,15}$	30	$1^{15}2^{15}$	$\phi_{30,15.}+\phi_{15,16}$
28	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
29	20	$2A_1$	3	$\phi_{20,20}$	20	1^{20}	$\phi_{20,20}$
30	6	A_1	2	$\phi_{6,25}$	6	1^6	$\phi_{6,25}$
31	1	0	1	$\phi_{1,36}$	1	1^1	$\phi_{1,36}$

2.5. E_7 : **simply connected.**

12 Block

Block	#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
12	0	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
12	1	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
12	2	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$

13 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
2	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
3	756	A_4	21	$\phi_{420,13}$	420	$1^{84}2^{336}$	$\phi_{420,13}+\phi_{336,14}$
1	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
4	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
5	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
6	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
11	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
9	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
10	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
12	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
15	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
18	77	A_2	6	$\phi_{56,30}$	56	$1^{35}2^{21}$	$\phi_{56,30}+\phi_{21,33}$
7	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
8	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
13	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
16	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
14	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
17	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
19	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
20	7	A_1	2	$\phi_{7,46}$	7	1^{7}	$\phi_{7,46}$
21	7	A_1	2	$\phi_{7,46}$	7	1^{7}	$\phi_{7,46}$
22	1	0	1	$\phi_{1,63}$	1	1^{1}	$\phi_{1,63}$

21 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
1	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
2	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$

23 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
1	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15,7}$
10	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
2	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
4	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
5	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
11	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
12	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
6	756	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{84}2^{336}$	$\phi_{420,10}+\phi_{336,11}$
13	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
7	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
14	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
15	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
3	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
8	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
9	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
19	665	$D_4(a_1)$	16	$\phi_{315,16}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,16}+\phi_{280,17}+\phi_{70,18}$
20	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
16	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
22	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
17	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
21	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
23	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
24	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
25	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
18	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
26	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
27	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
28	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
29	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
30	7	A_1	2	$\phi_{7,46}$	7	1^{7}	$\phi_{7,46}$
31	1	0	1	$\phi_{1,63}$	1	1^{1}	$\phi_{1,63}$

31 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^{1}	$\phi_{1,0}$
1	1	E_7	45	$\phi_{1,0}$	1	1^{1}	$\phi_{1,0}$
2	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^{7}	$\phi_{7,1}$
3	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
4	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
5	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
7	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
6	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
14	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
8	225	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4}+\phi_{105,5}$
11	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
9	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
10	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
12	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
15	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
13	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
16	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$

32 Block

#	$\|C\|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
2	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
3	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
4	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
5	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
13	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
6	225	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4}+\phi_{105,5}$
7	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
9	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
10	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
14	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
8	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
15	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
11	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$
12	665	$E_7(a_5)$	34	$\phi_{315,7}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,7}+\phi_{280,8}+\phi_{70,9}$
21	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$
16	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
17	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
23	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
18	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
22	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
29	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
19	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
24	756	A_4	21	$\phi_{420,13}$	420	$1^{84}2^{336}$	$\phi_{420,13}+\phi_{336,14}$
20	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
25	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
26	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
30	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
27	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
28	135	A_2+A_1	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25}+\phi_{15,28}$
31	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$

33 Block

#	$\|C\|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
2	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
3	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
4	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
6	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
5	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
7	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
10	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
11	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
8	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
9	225	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4}+\phi_{105,5}$
12	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15.7}$
13	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
14	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
15	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
18	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
22	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
16	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
24	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
17	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
23	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
19	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
27	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
20	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$
25	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$
29	665	$E_7(a_5)$	34	$\phi_{315,7}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,7}+\phi_{280,8}+\phi_{70,9}$
30	665	$E_7(a_5)$	34	$\phi_{315,7}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,7}+\phi_{280,8}+\phi_{70,9}$
26	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
28	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
31	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
21	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
32	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
33	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
35	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
42	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
36	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
37	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
34	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
39	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
44	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
38	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
40	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
45	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$

33 Block cont'd

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
52	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
53	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
41	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
43	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
46	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
49	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
54	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
57	756	A_4	21	$\phi_{420,13}$	420	$1^{84}2^{336}$	$\phi_{420,13}+\phi_{336,14}$
55	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
56	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
58	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
63	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
47	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
50	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
59	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
60	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
61	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
62	665	$D_4(a_1)$	16	$\phi_{315,16}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,16}+\phi_{280,17}+\phi_{70,18}$
64	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
69	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
70	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
48	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
51	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
65	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
67	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
71	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
72	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
66	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
68	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
73	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
74	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
75	135	A_2+A_1	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25}+\phi_{15,28}$
76	135	A_2+A_1	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25}+\phi_{15,28}$
78	135	A_2+A_1	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25}+\phi_{15,28}$
77	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
79	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
80	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
81	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
82	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
83	7	A_1	2	$\phi_{7,46}$	7	1^{7}	$\phi_{7,46}$
84	1	0	1	$\phi_{1,63}$	1	1^{1}	$\phi_{1,63}$

2.6. E_7 : **adjoint.**

12 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
1	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
2	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$

13 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
1	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
2	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
3	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
6	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
7	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
5	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
8	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
11	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
4	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
9	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
10	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
12	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
13	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
14	7	A_1	2	$\phi_{7,46}$	7	1^{7}	$\phi_{7,46}$
15	1	0	1	$\phi_{1,63}$	1	1^{1}	$\phi_{1,63}$
16	1	0	1	$\phi_{1,63}$	1	1^{1}	$\phi_{1,63}$

21 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
1	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
2	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$

23 Block

#	$\|C\|$	$\mathcal{O}_C$	n	$\sigma_{spec}\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	$\mathcal{P}^t_{\tau_\infty}$	W-rep
0	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
1	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15,7}$
10	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
2	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
4	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
5	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
11	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
12	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
6	756	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{84}2^{336}$	$\phi_{420,10}+\phi_{336,11}$
13	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
7	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
14	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
15	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
3	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
8	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
9	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
19	665	$D_4(a_1)$	16	$\phi_{315,16}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,16}+\phi_{280,17}+\phi_{70,18}$
20	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
16	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
22	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
17	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
21	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
23	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
24	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
25	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
18	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
26	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
27	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
28	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
29	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
30	7	A_1	2	$\phi_{7,46}$	7	1^{7}	$\phi_{7,46}$
31	1	0	1	$\phi_{1,63}$	1	1^{1}	$\phi_{1,63}$

31 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
2	7	$E_7(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
3	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
4	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
6	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
8	77	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{35}2^{21}$	$\phi_{56,3}+\phi_{21,6}$
5	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
7	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
17	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
18	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
9	225	$E_6\,(a_1)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4}+\phi_{105,5}$
10	225	$E_6\,(a_1)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4}+\phi_{105,5}$
12	210	$D_6\,(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
13	210	$D_6\,(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
11	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
14	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
15	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
19	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
20	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
16	756	$D_5\,(a_1)$	26	$\phi_{420,10}$	420	$1^{84}2^{336}$	$\phi_{420,10}+\phi_{336,11}$
21	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
22	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$

32 Block

#	$\|C\|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	7	$E_7\,(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
2	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
3	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
4	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
5	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
13	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
6	225	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{15}2^{105}$	$\phi_{120,4}+\phi_{105,5}$
7	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
9	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
10	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
14	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
8	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
15	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
11	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$
12	665	$E_7(a_5)$	34	$\phi_{315,7}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,7}+\phi_{280,8}+\phi_{70,9}$
21	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$
16	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
17	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
23	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
18	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
22	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
29	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
19	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
24	756	A_4	21	$\phi_{420,13}$	420	$1^{84}2^{336}$	$\phi_{420,13}+\phi_{336,14}$
20	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
25	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
26	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
30	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
27	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
28	135	A_2+A_1	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25}+\phi_{15,28}$
31	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$

33 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	7	$E_7\,(a_1)$	44	$\phi_{7,1}$	7	1^7	$\phi_{7,1}$
2	27	$E_7(a_2)$	43	$\phi_{27,2}$	27	1^{27}	$\phi_{27,2}$
3	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
4	91	$E_7(a_3)$	42	$\phi_{56,3}$	56	$1^{21}2^{35}$	$\phi_{56,3}+\phi_{35,4}$
5	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
7	21	E_6	41	$\phi_{21,3}$	21	1^{21}	$\phi_{21,3}$
6	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15,7}$
8	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15,7}$
10	135	$E_6(a_1)$	40	$\phi_{120,4}$	120	$1^{105}2^{15}$	$\phi_{120,4}+\phi_{15,7}$
9	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
11	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
14	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
16	189	$E_7(a_4)$	38	$\phi_{189,5}$	189	1^{189}	$\phi_{189,5}$
12	210	$D_6(a_1)$	37	$\phi_{210,6}$	210	1^{210}	$\phi_{210,6}$
13	168	D_5+A_1	36	$\phi_{168,6}$	168	1^{168}	$\phi_{168,6}$
15	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
17	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
21	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
24	105	A_6	35	$\phi_{105,6}$	105	1^{105}	$\phi_{105,6}$
18	875	$E_7(a_5)$	34	$\phi_{315,7}$	315	$2^{70}3^{245}$	$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$
26	665	$E_7(a_5)$	34	$\phi_{315,7}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,7}+\phi_{280,8}+\phi_{70,9}$
20	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
23	189	D_5	33	$\phi_{189,7}$	189	1^{189}	$\phi_{189,7}$
22	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
25	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
28	621	$E_6(a_3)$	32	$\phi_{405,8}$	405	$1^{189}2^{216}$	$\phi_{405,8}+\phi_{216,9}$
19	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
29	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
30	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
31	378	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	378	1^{378}	$\phi_{378,9}$
33	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
34	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
35	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
36	210	A_4+A_2	27	$\phi_{210,10}$	210	1^{210}	$\phi_{210,10}$
32	504	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{336}2^{84}$	$\phi_{420,10}+\phi_{84,12}$
38	756	$D_5(a_1)$	26	$\phi_{420,10}$	420	$1^{84}2^{336}$	$\phi_{420,10}+\phi_{336,11}$
37	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
39	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$
40	1024	A_4+A_1	25	$\phi_{512,11}$	512	2^{512}	$\phi_{512,11}+\phi_{512,12}$

33 Block cont'd

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
45	105	$(A_5)''$	23	$\phi_{105,12}$	105	1^{105}	$\phi_{105,12}$
41	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
44	210	$A_3+A_2+A_1$	22	$\phi_{210,13}$	210	1^{210}	$\phi_{210,13}$
46	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
47	504	A_4	21	$\phi_{420,13}$	420	$1^{336}2^{84}$	$\phi_{420,13}+\phi_{84,15}$
49	504	A_4	21	$\phi_{420,13}$	420	$1^{84}2^{336}$	$\phi_{420,13}+\phi_{84,15}$
48	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
50	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
51	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
57	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
59	378	A_3+A_2	20	$\phi_{378,14}$	378	1^{378}	$\phi_{378,14}$
42	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
52	621	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	405	$1^{189}2^{216}$	$\phi_{405,15}+\phi_{216,16}$
53	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
55	105	D_4	18	$\phi_{105,15}$	105	1^{105}	$\phi_{105,15}$
54	665	$D_4(a_1)$	16	$\phi_{315,16}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,16}+\phi_{280,17}+\phi_{70,18}$
56	665	$D_4(a_1)$	16	$\phi_{315,16}$	315	$1^{35}2^{210}3^{70}$	$\phi_{315,16}+\phi_{280,17}+\phi_{70,18}$
58	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
60	875	$D_4(a_1)$	16	$\phi_{315,16}$	315	$2^{70}3^{245}$	$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$
63	189	$(A_3+A_1)''$	13	$\phi_{189,20}$	189	1^{189}	$\phi_{189,20}$
43	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
61	105	A_2+3A_1	12	$\phi_{105,21}$	105	1^{105}	$\phi_{105,21}$
64	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
65	168	$2A_2$	11	$\phi_{168,21}$	168	1^{168}	$\phi_{168,21}$
66	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
69	210	A_3	10	$\phi_{210,21}$	210	1^{210}	$\phi_{210,21}$
62	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
67	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
68	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
70	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
71	189	A_2+2A_1	9	$\phi_{189,22}$	189	1^{189}	$\phi_{189,22}$
72	135	A_2+A_1	8	$\phi_{120,25}$	120	$1^{105}2^{15}$	$\phi_{120,25}+\phi_{15,28}$
75	225	A_2+A_1	8	$\phi_{120,25}$	120	$1^{15}2^{105}$	$\phi_{120,25}+\phi_{105,26}$
73	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
74	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
76	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
77	91	A_2	6	$\phi_{56,30}$	56	$1^{21}2^{35}$	$\phi_{56,30}+\phi_{35,31}$
78	21	$(3A_1)''$	4	$\phi_{21,36}$	21	1^{21}	$\phi_{21,36}$
79	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
80	27	$2A_1$	3	$\phi_{27,37}$	27	1^{27}	$\phi_{27,37}$
81	7	A_1	2	$\phi_{7,46}$	7	1^{7}	$\phi_{7,46}$
82	7	A_1	2	$\phi_{7,46}$	7	1^{7}	$\phi_{7,46}$
83	1	0	1	$\phi_{1,63}$	1	1^{1}	$\phi_{1,63}$
84	1	0	1	$\phi_{1,63}$	1	1^{1}	$\phi_{1,63}$

2.7. E_8.

11 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	525	E_6	50	$\phi_{525,12}$	525	1^{525}	$\phi_{525,12}$
1	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
2	525	D_4	15	$\phi_{525,36}$	525	1^{525}	$\phi_{525,36}$

12 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	525	E_6	50	$\phi_{525,12}$	525	1^{525}	$\phi_{525,12}$
1	3500	$E_6\,(a_1)$	47	$\phi_{2800,13}$	2800	$1^{2100}2^{700}$	$\phi_{2800,13}+\phi_{700,16}$
5	4200	A_6	43	$\phi_{4200,15}$	4200	1^{4200}	$\phi_{4200,15}$
2	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
3	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
4	8800	$E_6\,(a_3)$	33	$\phi_{5600,21}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,21}+\phi_{3200,22}$
6	8800	$E_6\,(a_3)$	33	$\phi_{5600,21}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,21}+\phi_{3200,22}$
7	4536	A_4+A_2	28	$\phi_{4536,23}$	4536	1^{4536}	$\phi_{4536,23}$
8	4900	$D_5\,(a_1)$	26	$\phi_{2800,25}$	2800	$1^{700}2^{2100}$	$\phi_{2800,25}+\phi_{2100,28}$
9	8192	A_4+A_1	24	$\phi_{4096,26}$	4096	2^{4096}	$\phi_{4096,26}+\phi_{4096,27}$
11	3240	A_4	20	$\phi_{2268,30}$	2268	$1^{1296}2^{972}$	$\phi_{2268,30}+\phi_{972,32}$
14	3240	A_4	20	$\phi_{2268,30}$	2268	$1^{1296}2^{972}$	$\phi_{2268,30}+\phi_{972,32}$
18	3240	A_3+A_2	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
10	525	D_4	15	$\phi_{525,36}$	525	1^{525}	$\phi_{525,36}$
13	525	D_4	15	$\phi_{525,36}$	525	1^{525}	$\phi_{525,36}$
12	3192	$D_4\,(a_1)$	14	$\phi_{1400,37}$	1400	$1^{56}2^{896}3^{448}$	$\phi_{1400,37}+\phi_{1344,38}+\phi_{448,39}$
15	3752	$D_4\,(a_1)$	14	$\phi_{1400,37}$	1400	$2^{448}3^{952}$	$\phi_{1400,37}+\phi_{1344,38}+\phi_{1008,39}$
19	3752	$D_4\,(a_1)$	14	$\phi_{1400,37}$	1400	$2^{448}3^{952}$	$\phi_{1400,37}+\phi_{1344,38}+\phi_{1008,39}$
20	1000	$2A_2$	11	$\phi_{700,42}$	700	$1^{400}2^{300}$	$\phi_{700,42}+\phi_{300,44}$
16	567	A_3	9	$\phi_{567,46}$	567	1^{567}	$\phi_{567,46}$
21	567	A_3	9	$\phi_{567,46}$	567	1^{567}	$\phi_{567,46}$
22	560	A_2+2A_1	8	$\phi_{560,47}$	560	1^{560}	$\phi_{560,47}$
24	560	A_2+2A_1	8	$\phi_{560,47}$	560	1^{560}	$\phi_{560,47}$
25	370	A_2+A_1	7	$\phi_{210,52}$	210	$1^{50}2^{160}$	$\phi_{210,52}+\phi_{160,55}$
17	196	A_2	5	$\phi_{112,63}$	112	$1^{28}2^{84}$	$\phi_{112,63}+\phi_{84,64}$
23	196	A_2	5	$\phi_{112,63}$	112	$1^{28}2^{84}$	$\phi_{112,63}+\phi_{84,64}$
26	196	A_2	5	$\phi_{112,63}$	112	$1^{28}2^{84}$	$\phi_{112,63}+\phi_{84,64}$
27	35	$2A_1$	3	$\phi_{35,74}$	35	1^{35}	$\phi_{35,74}$
28	35	$2A_1$	3	$\phi_{35,74}$	35	1^{35}	$\phi_{35,74}$
29	8	A_1	2	$\phi_{8,91}$	8	1^{8}	$\phi_{8,91}$
30	1	1	1	$\phi_{1,120}$	1	1^{1}	$\phi_{1,120}$

21 Block

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_8	70	$\phi_{1,0}$	1	1^1	$\phi_{1,0}$
1	8	$E_8\,(a_1)$	69	$\phi_{8,1}$	8	1^8	$\phi_{8,1}$
2	35	$E_8\,(a_2)$	68	$\phi_{35,2}$	35	1^{35}	$\phi_{35,2}$
3	35	$E_8\,(a_2)$	68	$\phi_{35,2}$	35	1^{35}	$\phi_{35,2}$
4	196	$E_8\,(a_3)$	67	$\phi_{112,3}$	112	$1^{28}2^{84}$	$\phi_{112,3}+\phi_{84,4}$
5	196	$E_8\,(a_3)$	67	$\phi_{112,3}$	112	$1^{28}2^{84}$	$\phi_{112,3}+\phi_{84,4}$
13	196	$E_8\,(a_3)$	67	$\phi_{112,3}$	112	$1^{28}2^{84}$	$\phi_{112,3}+\phi_{84,4}$
6	370	$E_8\,(a_4)$	66	$\phi_{210,4}$	210	$1^{50}2^{160}$	$\phi_{210,4}+\phi_{160,7}$
7	560	$E_8\,(b_4)$	64	$\phi_{560,5}$	560	1^{560}	$\phi_{560,5}$
9	560	$E_8\,(b_4)$	64	$\phi_{560,5}$	560	1^{560}	$\phi_{560,5}$
8	1000	$E_8\,(a_5)$	63	$\phi_{700,6}$	700	$1^{400}2^{300}$	$\phi_{700,6}+\phi_{300,8}$
10	567	$E_7\,(a_1)$	62	$\phi_{567,6}$	567	1^{567}	$\phi_{567,6}$
14	567	$E_7\,(a_1)$	62	$\phi_{567,6}$	567	1^{567}	$\phi_{567,6}$
11	3752	$E_8\,(b_5)$	61	$\phi_{1400,7}$	1400	$2^{448}3^{952}$	$\phi_{1400,7}+\phi_{1344,8}+\phi_{1008,9}$
12	3192	$E_8\,(b_5)$	61	$\phi_{1400,7}$	1400	$1^{56}2^{896}3^{448}$	$\phi_{1400,7}+\phi_{1344,8}+\phi_{448,9}$
20	3752	$E_8\,(b_5)$	61	$\phi_{1400,7}$	1400	$2^{448}3^{952}$	$\phi_{1400,7}+\phi_{1344,8}+\phi_{1008,9}$
15	3240	$D_7\,(a_1)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
16	3240	$E_7\,(a_3)$	54	$\phi_{2268,10}$	2268	$1^{1296}2^{972}$	$\phi_{2268,10}+\phi_{972,12}$
22	3240	$E_7\,(a_3)$	54	$\phi_{2268,10}$	2268	$1^{1296}2^{972}$	$\phi_{2268,10}+\phi_{972,12}$
17	8192	$E_6\,(a_1)+A_1$	53	$\phi_{4096,11}$	4096	2^{4096}	$\phi_{4096,11}+\phi_{4096,12}$
21	525	E_6	50	$\phi_{525,12}$	525	1^{525}	$\phi_{525,12}$
28	525	E_6	50	$\phi_{525,12}$	525	1^{525}	$\phi_{525,12}$
18	4536	D_5+A_2	48	$\phi_{4536,13}$	4536	1^{4536}	$\phi_{4536,13}$
23	4900	$E_6\,(a_1)$	47	$\phi_{2800,13}$	2800	$1^{700}2^{2100}$	$\phi_{2800,13}+\phi_{2100,16}$
19	8800	$D_6\,(a_1)$	44	$\phi_{5600,15}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,15}+\phi_{3200,16}$
24	8800	$D_6\,(a_1)$	44	$\phi_{5600,15}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,15}+\phi_{3200,16}$
25	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
29	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
26	4200	D_4+A_2	32	$\phi_{4200,21}$	4200	1^{4200}	$\phi_{4200,21}$
27	3500	$D_5\,(a_1)$	26	$\phi_{2800,25}$	2800	$1^{2100}2^{700}$	$\phi_{2800,25}+\phi_{700,28}$
30	525	D_4	15	$\phi_{525,36}$	525	1^{525}	$\phi_{525,36}$

22 Block

#	$\|C\|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_8	70	$\phi_{1,0}$	1	1^{1}	$\phi_{1,0}$
1	8	$E_8\,(a_1)$	69	$\phi_{8,1}$	8	1^{8}	$\phi_{8,1}$
2	35	$E_8\,(a_2)$	68	$\phi_{35,2}$	35	1^{35}	$\phi_{35,2}$
3	196	$E_8\,(a_3)$	67	$\phi_{112,3}$	112	$1^{28}2^{84}$	$\phi_{112,3}+\phi_{84.4}$
5	196	$E_8\,(a_3)$	67	$\phi_{112,3}$	112	$1^{28}2^{84}$	$\phi_{112,3}+\phi_{84.4}$
4	260	$E_8\,(a_4)$	66	$\phi_{210,4}$	210	$1^{160}2^{50}$	$\phi_{210,4}+\phi_{50,8}$
7	260	$E_8\,(a_4)$	66	$\phi_{210,4}$	210	$1^{160}2^{50}$	$\phi_{210,4}+\phi_{50,8}$
6	560	$E_8\,(b_4)$	64	$\phi_{560,5}$	560	1^{560}	$\phi_{560,5}$
8	560	$E_8\,(b_4)$	64	$\phi_{560,5}$	560	1^{560}	$\phi_{560,5}$
9	560	$E_8\,(b_4)$	64	$\phi_{560,5}$	560	1^{560}	$\phi_{560,5}$
10	1100	$E_8\,(a_5)$	63	$\phi_{700,6}$	700	$1^{300}2^{400}$	$\phi_{700,6}+\phi_{400,7}$
13	1100	$E_8\,(a_5)$	63	$\phi_{700,6}$	700	$1^{300}2^{400}$	$\phi_{700,6}+\phi_{400,7}$
17	1100	$E_8\,(a_5)$	63	$\phi_{700,6}$	700	$1^{300}2^{400}$	$\phi_{700,6}+\phi_{400,7}$
11	567	$E_7\,(a_1)$	62	$\phi_{567,6}$	567	1^{567}	$\phi_{567,6}$
12	3192	$E_8\,(b_5)$	61	$\phi_{1400,7}$	1400	$1^{56}2^{896}3^{448}$	$\phi_{1400,7}+\phi_{1344,8}+\phi_{448,9}$
14	3752	$E_8\,(b_5)$	61	$\phi_{1400,7}$	1400	$2^{448}3^{952}$	$\phi_{1400,7}+\phi_{1344,8}+\phi_{1008,9}$
15	4025	$E_8\,(a_6)$	59	$\phi_{1400,8}$	1400	$2^{175}3^{1225}$	$\phi_{1400,8}+\phi_{1575,10}+\phi_{1050,10}$
18	2625	$E_8\,(a_6)$	59	$\phi_{1400,8}$	1400	$1^{350}2^{875}3^{175}$	$\phi_{1400,8}+\phi_{1050,10}+\phi_{175,12}$
16	3240	$D_7\,(a_1)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
19	3240	$D_7\,(a_1)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
20	3240	$D_7\,(a_1)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
21	3240	$D_7\,(a_1)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
22	3240	$D_7\,(a_1)$	56	$\phi_{3240,9}$	3240	1^{3240}	$\phi_{3240,9}$
23	3640	$E_8\,(b_6)$	55	$\phi_{2240,10}$	2240	$1^{840}2^{1400}$	$\phi_{2240,10}+\phi_{1400,11}$
25	3640	$E_8\,(b_6)$	55	$\phi_{2240,10}$	2240	$1^{840}2^{1400}$	$\phi_{2240,10}+\phi_{1400,11}$
27	3640	$E_8\,(b_6)$	55	$\phi_{2240,10}$	2240	$1^{840}2^{1400}$	$\phi_{2240,10}+\phi_{1400,11}$
24	3240	$E_7\,(a_3)$	54	$\phi_{2268,10}$	2268	$1^{1296}2^{972}$	$\phi_{2268,10}+\phi_{972,12}$
29	3240	$E_7\,(a_3)$	54	$\phi_{2268,10}$	2268	$1^{1296}2^{972}$	$\phi_{2268,10}+\phi_{972,12}$
26	8192	$E_6\,(a_1)+A_1$	53	$\phi_{4096,11}$	4096	2^{4096}	$\phi_{4096,11}+\phi_{4096,12}$
30	8192	$E_6\,(a_1)+A_1$	53	$\phi_{4096,11}$	4096	2^{4096}	$\phi_{4096,11}+\phi_{4096,12}$
28	7560	$D_7\,(a_2)$	51	$\phi_{4200,12}$	4200	$1^{840}2^{3360}$	$\phi_{4200,12}+\phi_{3360,13}$
31	5040	$D_7\,(a_2)$	51	$\phi_{4200,12}$	4200	$1^{3360}2^{840}$	$\phi_{4200,12}+\phi_{840,14}$
35	7560	$D_7\,(a_2)$	51	$\phi_{4200,12}$	4200	$1^{840}2^{3360}$	$\phi_{4200,12}+\phi_{3360,13}$
33	525	E_6	50	$\phi_{525,12}$	525	1^{525}	$\phi_{525,12}$
32	4536	D_5+A_2	48	$\phi_{4536,13}$	4536	1^{4536}	$\phi_{4536,13}$
38	4536	D_5+A_2	48	$\phi_{4536,13}$	4536	1^{4536}	$\phi_{4536,13}$
39	4536	D_5+A_2	48	$\phi_{4536,13}$	4536	1^{4536}	$\phi_{4536,13}$
45	4536	D_5+A_2	48	$\phi_{4536,13}$	4536	1^{4536}	$\phi_{4536,13}$
34	3500	$E_6\,(a_1)$	47	$\phi_{2800,13}$	2800	$1^{2100}2^{700}$	$\phi_{2800,13}+\phi_{700,16}$
37	3500	$E_6\,(a_1)$	47	$\phi_{2800,13}$	2800	$1^{2100}2^{700}$	$\phi_{2800,13}+\phi_{700,16}$
36	6075	$E_7\,(a_4)$	46	$\phi_{6075,14}$	6075	1^{6075}	$\phi_{6075,14}$
41	6075	$E_7\,(a_4)$	46	$\phi_{6075,14}$	6075	1^{6075}	$\phi_{6075,14}$
43	6075	$E_7\,(a_4)$	46	$\phi_{6075,14}$	6075	1^{6075}	$\phi_{6075,14}$
40	2835	A_6+A_1	45	$\phi_{2835,14}$	2835	1^{2835}	$\phi_{2835,14}$
44	8800	$D_6\,(a_1)$	44	$\phi_{5600,15}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,15}+\phi_{3200,16}$
49	8800	$D_6\,(a_1)$	44	$\phi_{5600,15}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,15}+\phi_{3200,16}$
42	4200	A_6	43	$\phi_{4200,15}$	4200	1^{4200}	$\phi_{4200,15}$
46	4200	A_6	43	$\phi_{4200,15}$	4200	1^{4200}	$\phi_{4200,15}$
48	4200	A_6	43	$\phi_{4200,15}$	4200	1^{4200}	$\phi_{4200,15}$
50	4200	A_6	43	$\phi_{4200,15}$	4200	1^{4200}	$\phi_{4200,15}$
47	22778	$E_8\,(a_7)$	42	$\phi_{4480,16}$	4480	$2^{70}3^{378}4^{1092}$ $\cdot5^{924}6^{1596}7^{420}$	$\phi_{4480,16}+\phi_{7168,17}+\phi_{3150,18}$ $+\phi_{4200,18}+\phi_{1344,19}+\phi_{2016,19}$ $+\phi_{420,20}$
51	38766	$E_8\,(a_7)$	42	$\phi_{4480,16}$	4480	$5^{420}7^{1134}$ $\cdot8^{1596}12^{1330}$	$\phi_{4480,16}+\phi_{7168,17}+2\phi_{4200,18}$ $+\phi_{4536,18}+\phi_{5670,18}+\phi_{1344,19}$ $+\phi_{5600,19}+\phi_{1400,20}+\phi_{168,24}$
53	46676	$E_8\,(a_7)$	42	$\phi_{4480,16}$	4480	$6^{420}7^{756}8^{168}$ $\cdot9^{378}11^{70}$ $\cdot12^{1596}13^{1092}$	$\phi_{4480,16}+2\phi_{7168,17}+\phi_{3150,18}$ $+\phi_{4200,18}+\phi_{4536,18}+\phi_{5670,18}$ $+\phi_{2016,19}+\phi_{5600,19}+\phi_{2688,20}$

#	$\|C\|$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
54	2100	D_5	34	$\phi_{2100,20}$	2100	1^{2100}	$\phi_{2100,20}$
55	8800	$E_6(a_3)$	33	$\phi_{5600,21}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,21}+\phi_{3200,22}$
63	8800	$E_6(a_3)$	33	$\phi_{5600,21}$	5600	$1^{2400}2^{3200}$	$\phi_{5600,21}+\phi_{3200,22}$
52	4200	D_4+A_2	32	$\phi_{4200,21}$	4200	1^{4200}	$\phi_{4200,21}$
57	4200	D_4+A_2	32	$\phi_{4200,21}$	4200	1^{4200}	$\phi_{4200,21}$
59	4200	D_4+A_2	32	$\phi_{4200,21}$	4200	1^{4200}	$\phi_{4200,21}$
62	4200	D_4+A_2	32	$\phi_{4200,21}$	4200	1^{4200}	$\phi_{4200,21}$
60	2835	$A_4+A_2+A_1$	31	$\phi_{2835,22}$	2835	1^{2835}	$\phi_{2835,22}$
58	6075	$D_5(a_1)+A_1$	30	$\phi_{6075,22}$	6075	1^{6075}	$\phi_{6075,22}$
64	6075	$D_5(a_1)+A_1$	30	$\phi_{6075,22}$	6075	1^{6075}	$\phi_{6075,22}$
65	6075	$D_5(a_1)+A_1$	30	$\phi_{6075,22}$	6075	1^{6075}	$\phi_{6075,22}$
56	4536	A_4+A_2	28	$\phi_{4536,23}$	4536	1^{4536}	$\phi_{4536,23}$
66	4536	A_4+A_2	28	$\phi_{4536,23}$	4536	1^{4536}	$\phi_{4536,23}$
68	4536	A_4+A_2	28	$\phi_{4536,23}$	4536	1^{4536}	$\phi_{4536,23}$
69	4536	A_4+A_2	28	$\phi_{4536,23}$	4536	1^{4536}	$\phi_{4536,23}$
61	7560	A_4+2A_1	27	$\phi_{4200,24}$	4200	$1^{840}2^{3360}$	$\phi_{4200,24}+\phi_{3360,25}$
67	7560	A_4+2A_1	27	$\phi_{4200,24}$	4200	$1^{840}2^{3360}$	$\phi_{4200,24}+\phi_{3360,25}$
70	5040	A_4+2A_1	27	$\phi_{4200,24}$	4200	$1^{3360}2^{840}$	$\phi_{4200,24}+\phi_{840,26}$
71	3500	$D_5(a_1)$	26	$\phi_{2800,25}$	2800	$1^{2100}2^{700}$	$\phi_{2800,25}+\phi_{700,28}$
79	3500	$D_5(a_1)$	26	$\phi_{2800,25}$	2800	$1^{2100}2^{700}$	$\phi_{2800,25}+\phi_{700,28}$
72	8192	A_4+A_1	24	$\phi_{4096,26}$	4096	2^{4096}	$\phi_{4096,26}+\phi_{4096,27}$
80	8192	A_4+A_1	24	$\phi_{4096,26}$	4096	2^{4096}	$\phi_{4096,26}+\phi_{4096,27}$
73	3640	$D_4(a_1)+A_2$	23	$\phi_{2240,28}$	2240	$1^{840}2^{1400}$	$\phi_{2240,28}+\phi_{1400,29}$
78	3640	$D_4(a_1)+A_2$	23	$\phi_{2240,28}$	2240	$1^{840}2^{1400}$	$\phi_{2240,28}+\phi_{1400,29}$
81	3640	$D_4(a_1)+A_2$	23	$\phi_{2240,28}$	2240	$1^{840}2^{1400}$	$\phi_{2240,28}+\phi_{1400,29}$
74	3240	A_4	20	$\phi_{2268,30}$	2268	$1^{1296}2^{972}$	$\phi_{2268,30}+\phi_{972,32}$
82	3240	A_4	20	$\phi_{2268,30}$	2268	$1^{1296}2^{972}$	$\phi_{2268,30}+\phi_{972,32}$
75	3240	A_3+A_2	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
76	3240	A_3+A_2	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
83	3240	A_3+A_2	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
84	3240	A_3+A_2	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
86	3240	A_3+A_2	19	$\phi_{3240,31}$	3240	1^{3240}	$\phi_{3240,31}$
87	4025	$D_4(a_1)+A_1$	18	$\phi_{1400,32}$	1400	$2^{175}3^{1225}$	$\phi_{1400,32}+\phi_{1575,34}+\phi_{1050,34}$
89	2625	$D_4(a_1)+A_1$	18	$\phi_{1400,32}$	1400	$1^{350}2^{875}3^{175}$	$\phi_{1400,32}+\phi_{1050,34}+\phi_{175,36}$
85	525	D_4	15	$\phi_{525,36}$	525	1^{525}	$\phi_{525,36}$
88	3752	$D_4(a_1)$	14	$\phi_{1400,37}$	1400	$2^{448}3^{952}$	$\phi_{1400,37}+\phi_{1344,38}+\phi_{1008,39}$
90	3192	$D_4(a_1)$	14	$\phi_{1400,37}$	1400	$1^{56}2^{896}3^{448}$	$\phi_{1400,37}+\phi_{1344,38}+\phi_{448,39}$
77	1100	$2A_2$	11	$\phi_{700,42}$	700	$1^{300}2^{400}$	$\phi_{700,42}+\phi_{400,43}$
91	1100	$2A_2$	11	$\phi_{700,42}$	700	$1^{300}2^{400}$	$\phi_{700,42}+\phi_{400,43}$
92	1100	$2A_2$	11	$\phi_{700,42}$	700	$1^{300}2^{400}$	$\phi_{700,42}+\phi_{400,43}$
94	567	A_3	9	$\phi_{567,46}$	567	1^{567}	$\phi_{567,46}$
93	560	A_2+2A_1	8	$\phi_{560,47}$	560	1^{560}	$\phi_{560,47}$
95	560	A_2+2A_1	8	$\phi_{560,47}$	560	1^{560}	$\phi_{560,47}$
96	560	A_2+2A_1	8	$\phi_{560,47}$	560	1^{560}	$\phi_{560,47}$
97	260	A_2+A_1	7	$\phi_{210,52}$	210	$1^{160}2^{50}$	$\phi_{210,52}+\phi_{50,56}$
99	260	A_2+A_1	7	$\phi_{210,52}$	210	$1^{160}2^{50}$	$\phi_{210,52}+\phi_{50,56}$
98	196	A_2	5	$\phi_{112,63}$	112	$1^{28}2^{84}$	$\phi_{112,63}+\phi_{84,64}$
100	196	A_2	5	$\phi_{112,63}$	112	$1^{28}2^{84}$	$\phi_{112,63}+\phi_{84,64}$
101	35	$2A_1$	3	$\phi_{35,74}$	35	1^{35}	$\phi_{35,74}$
102	8	A_1	2	$\phi_{8,91}$	8	1^{8}	$\phi_{8,91}$
103	1	0	1	$\phi_{1,120}$	1	1^{1}	$\phi_{1,120}$

2.8. $G_2\,(\mathbb{C})$.

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	G_2	5	$\phi_{1,0}$	1	1^1	$(\phi_{1,0})^2$
2	10	$G_2(a_1)$	4	$\phi_{2,1}$	4	$2^2 3^2$	$(\phi_{2,1})^2 + (\phi_{2,2})^2 + (\phi'_{1,3})^2 + (\phi''_{1,3})^2$
3	1	0	1	$\phi_{1,6}$	1	1^1	$(\phi_{1,6})^2$

2.9. $F_4\,(\mathbb{C})$.

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	F_4	16	$\phi_{1,0}$	1	1^1	$(\phi_{1,0})^2$
1	24	$F_4(a_1)$	15	$\phi_{4,1}$	16	$1^8 2^8$	$(\phi_{4,1})^2 + (\phi'_{2,4})^2 + (\phi''_{2,4})^2$
2	81	$F_4(a_2)$	14	$\phi_{9,2}$	81	1^{81}	$(\phi_{9,2})^2$
3	64	B_3	12	$\phi''_{8,3}$	64	1^{64}	$(\phi''_{8,3})^2$
4	64	C_3	13	$\phi'_{8,3}$	64	1^{64}	$(\phi'_{8,3})^2$
5	684	$F_4(a_3)$	11	$\phi_{12,4}$	144	$2^{12}3^{20}4^{48}5^{30}6^{16}9^{18}$	$(\phi_{12,4})^2 + (\phi_{16,5})^2 + (\phi'_{6,6})^2 + (\phi''_{6,6})^2 + (\phi'_{9,6})^2 + (\phi''_{9,6})^2 + (\phi'_{4,7})^2 + (\phi''_{4,7})^2 + (\phi_{4,8})^2 + (\phi'_{1,12})^2 + (\phi''_{1,12})^2$
6	64	A_2	5	$\phi''_{8,9}$	64	1^{64}	$(\phi''_{8,9})^2$
7	64	$\widetilde{A}_2$	6	$\phi'_{8,9}$	64	1^{64}	$(\phi'_{8,9})^2$
8	81	$A_1 + \tilde{A}_1$	4	$\phi_{9,10}$	81	1^{81}	$(\phi_{9,10})^2$
9	24	$\tilde{A}_1$	3	$\phi_{4,13}$	16	$1^8 2^8$	$(\phi_{4,13})^2 + (\phi'_{2,16})^2 + (\phi''_{2,16})^2$
10	1	0	1	$\phi_{1,24}$	1	1^1	$(\phi_{1,24})^2$

2.10. $E_6\,(\mathbb{C})$.

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim\,(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_6	21	$\phi_{1,0}$	1	1^1	$(\phi_{1,0})^2$
1	36	$E_6(a_1)$	20	$\phi_{6,1}$	36	1^{36}	$(\phi_{6,1})^2$
2	400	D_5	19	$\phi_{20,2}$	400	1^{400}	$(\phi_{20,2})^2$
3	1350	$E_6\,(a_3)$	18	$\phi_{30,3}$	900	$1^{450}2^{450}$	$(\phi_{30,3})^2 + (\phi_{15,4})^2 + (\phi_{15,5})^2$
4	4096	$D_5(a_1)$	17	$\phi_{64,4}$	4096	1^{4096}	$(\phi_{64,4})^2$
5	3600	$A_4 + A_1$	15	$\phi_{60,5}$	3600	1^{3600}	$(\phi_{60,5})^2$
6	6561	A_4	13	$\phi_{81,6}$	6561	1^{6561}	$(\phi_{81,6})^2$
7	576	D_4	13	$\phi_{24,6}$	576	1^{576}	$(\phi_{24,6})^2$
8	18600	$D_4(a_1)$	12	$\phi_{80,7}$	6400	$1^{400}2^{1000}3^{4600}6^{400}$	$(\phi_{80,7})^2 + (\phi_{90,8})^2 + (\phi_{60,8})^2 + (\phi_{10,9})^2 + (\phi_{20,10})^2$
9	6561	$A_3 + A_1$	9	$\phi_{81,10}$	6561	1^{6561}	$(\phi_{81,10})^2$
10	3600	$A_2 + 2A_1$	8	$\phi_{60,11}$	3600	1^{3600}	$(\phi_{60,11})^2$
11	576	$2A_2$	7	$\phi_{24,12}$	576	1^{576}	$(\phi_{24,12})^2$
12	4096	$A_2 + A_1$	6	$\phi_{64,13}$	4096	1^{4096}	$(\phi_{64,13})^2$
13	1350	A_2	5	$\phi_{30,15}$	900	$1^{450}2^{450}$	$(\phi_{30,15})^2 + (\phi_{15,16})^2 + (\phi_{15,17})^2$
14	400	$2A_1$	3	$\phi_{20,20}$	400	1^{400}	$(\phi_{20,20})^2$
15	36	A_1	2	$\phi_{6,25}$	36	1^{36}	$(\phi_{6,25})^2$
16	1	0	1	$\phi_{1,36}$	1	1^1	$(\phi_{1,36})^2$

2.11. $E_7(\mathbb{C})$.

#	$\lvert C\rvert$	$\mathcal{O}_C$	n	σ_{spec}	$\#Prim(C)$	$\mathcal{P}_{\tau_\infty}$	W-rep
0	1	E_7	45	$\phi_{1,0}$	1	1^1	$(\phi_{1,0})^2$
1	49	$E_7(a_1)$	44	$\phi_{7,1}$	49	1^{49}	$(\phi_{7,1})^2$
2	729	$E_7(a_2)$	43	$\phi_{27,2}$	729	1^{729}	$(\phi_{27,2})^2$
3	4802	$E_7(a_3)$	42	$\phi_{56,3}$	3136	$1^{1470}2^{1666}$	$(\phi_{56,3})^2+(\phi_{35,4})^2+(\phi_{21,6})^2$
4	441	E_6	41	$\phi_{21,3}$	441	1^{441}	$(\phi_{21,3})^2$
5	25650	$E_6(a_1)$	40	$\phi_{120,4}$	14400	$1^{3150}2^{11250}$	$(\phi_{120,4})^2+(\phi_{105,5})^2+(\phi_{15,7})^2$
6	35721	$E_7(a_4)$	38	$\phi_{189,5}$	35721	1^{35721}	$(\phi_{189,5})^2$
7	44100	$D_6(a_1)$	37	$\phi_{210,6}$	44100	1^{44100}	$(\phi_{210,6})^2$
8	28224	D_5+A_1	36	$\phi_{168,6}$	28224	1^{28224}	$(\phi_{168,6})^2$
9	11025	A_6	35	$\phi_{105,6}$	11025	1^{11025}	$(\phi_{105,6})^2$
10	262150	$E_7(a_5)$	34	$\phi_{315,7}$	99225	$1^{4900}2^{29400}3^{63700}6^{1225}$	$(\phi_{315,7})^2+(\phi_{280,8})^2+(\phi_{280,9})^2+(\phi_{35,13})^2$
11	35721	D_5	33	$\phi_{189,7}$	35721	1^{35721}	$(\phi_{189,7})^2$
12	246402	$E_6(a_3)$	32	$\phi_{405,8}$	164025	$1^{81648}2^{82377}$	$(\phi_{405,8})^2+(\phi_{216,9})^2+(\phi_{189,10})^2$
13	142884	$D_5(a_1)+A_1$	30	$\phi_{378,9}$	142884	1^{142884}	$(\phi_{378,9})^2$
14	296352	$D_5(a_1)$	26	$\phi_{420,10}$	176400	$1^{56448}2^{119952}$	$(\phi_{210,10})^2+(\phi_{336,11})^2+(\phi_{84,12})^2$
15	44100	A_4+A_2	26	$\phi_{210,10}$	44100	1^{44100}	$(\phi_{210,10})^2$
16	524288	A_4+A_1	25	$\phi_{512,11}$	262144	1^{262144}	$(\phi_{512,11})^2$
17	11025	$(A_5)''$	23	$\phi_{105,12}$	11025	1^{11025}	$(\phi_{105,12})^2$
18	296352	A_4	21	$\phi_{420,13}$	176400	$1^{56448}2^{119952}$	$(\phi_{420,13})^2+(\phi_{336,14})^2+(\phi_{84,15})^2$
19	44100	$A_3+A_2+A_1$	22	$\phi_{210,13}$	44100	1^{44100}	$(\phi_{210,13})^2$
20	142884	A_3+A_2	20	$\phi_{378,14}$	142884	1^{142884}	$(\phi_{378,14})^2$
21	246402	$D_4(a_1)+A_1$	19	$\phi_{405,15}$	164025	$1^{81648}2^{82377}$	$(\phi_{405,15})^2+(\phi_{216,16})^2+(\phi_{189,17})^2$
22	11025	D_4	18	$\phi_{105,15}$	11025	1^{11025}	$(\phi_{105,15})^2$
23	262150	$D_4(a_1)$	16	$\phi_{315,16}$	99225	$1^{4900}2^{29400}3^{63700}6^{1225}$	$(\phi_{315,16})^2+(\phi_{280,17})^2+(\phi_{280,18})^2+(\phi_{70,,18})^2+(\phi_{35,22})^2$
24	35721	$(A_3+A_1)''$	13	$\phi_{189,20}$	35721	1^{35721}	$(\phi_{189,20})^2$
25	11025	A_2+3A_1	12	$\phi_{105,21}$	11025	1^{11025}	$(\phi_{105,21})^2$
26	28224	$2A_2$	11	$\phi_{168,21}$	28224	1^{28224}	$(\phi_{168,21})^2$
27	44199	A_3	10	$\phi_{210,21}$	44100	1^{44100}	$(\phi_{210,21})^2$
28	35721	A_2+2A_1	9	$\phi_{189,22}$	35721	1^{35721}	$(\phi_{189,22})^2$
29	25650	A_2+A_1	8	$\phi_{120,25}$	14400	1^{14400}	$(\phi_{120,25})^2$
30	4802	A_2	6	$\phi_{56,30}$	3136	1^{3136}	$(\phi_{56,30})^2$
31	441	$(3A_1)''$	4	$\phi_{21,36}$	441	1^{441}	$(\phi_{21,36})^2$
32	729	$2A_1$	3	$\phi_{27,37}$	721	1^{729}	$(\phi_{27,37})^2$
33	49	A_1	2	$\phi_{7,46}$	49	1^{49}	$(\phi_{7,46})^2$
34	1	0	1	$\phi_{1,63}$	1	1^1	$(\phi_{1,63})^2$