Parameters

`p=(x,lambda,nu)`

$x \in K\backslash G/B \rightarrow \theta_x$

$\lambda \in X^* + \rho/(1-\theta_x)X^*$

$\nu \in X^*_{\mathbb{Q}}/(1+\theta_x)X^*_{\mathbb{Q}} \simeq (X^*_{\mathbb{Q}})^{-\theta_x}$

infinitesimal character

$$\begin{aligned}\gamma &= \frac{1+\theta_x}{2}\lambda + \frac{1-\theta_x}{2}\nu \\ &= \frac{1+\theta_x}{2}\lambda + \nu\end{aligned}$$

Parameters

Roughly: $x \to G(\mathbb{R})$-conjugacy class of Cartan subgroups H

θ_x=Cartan involution of $H(\mathbb{R})$

Last time:
$\theta_x = -Id \to H(\mathbb{R}) = \mathbb{R}^{*n}$ split

$\lambda \in X^*/2X^* \to$ character of $H(\mathbb{R})^{\theta_x} = (\mathbb{Z}/2\mathbb{Z})^n$

$\nu \in X^*_{\mathbb{Q}}$

$\mathrm{Ind}_B^G(\chi \otimes \nu)$

$\chi = \rho - \lambda$:, i.e.

$$\chi(\exp(2\pi i \mu^\vee)) = \exp(2\pi i \langle \rho - \lambda, \mu^\vee \rangle) \quad (\mu^\vee \in \frac{1}{2} X_*)$$

Next: $\theta_x = 1 \to$ discrete series

KGB

Working assumption today: G has discrete series representations

Given: $G = G(\mathbb{C})$ (assumption: distinguished involution $\delta = \mathrm{Id}$)

Always fixed fixed fixed: Cartan H $\subset$ Borel B

(H is "diagonal" and B is "upper triangular")

Fix $x_b \in G$, $x_b^2 \in Z(G) \rightarrow \theta = \mathrm{int}(x_b) \rightarrow$ real form of G, $K = G^{\theta_x}$ (complexified maximal compact subgroup of $G(\mathbb{R})$)

Study:

$$\begin{aligned} \{K\backslash G/B\} &= \{\text{K-orbits on the flag variety } G/B\} \\ &= \{\text{K-conjugacy classes of Borel subgroups of } G\} \end{aligned}$$

KGB

Theorem: $K\backslash G/B \leftrightarrow \{x \in \mathrm{Norm}_G(H) \mid x \sim_G x_b\}/H$

$\leftarrow$: if $x = gx_bg^{-1}$, then x goes to $[g^{-1}Bg]$, the K-conjugacy class of $g^{-1}Bg$

Note: $x = (gk)x_b(gk)^{-1} \rightarrow [k^{-1}(g^{-1}Bg)k] = [g^{-1}Bg]$

Definition:

$$\boxed{\mathcal{X} = \{x \in \mathrm{Norm}_G(H) \mid x \sim_g x_b\}/H}$$

(Really: $\mathcal{X}[x_b]$)

Atlas Command: `KGB` $\rightarrow \{x_0, \ldots, x_{n-1}\}$

Example: $SL(2, \mathbb{R})$

$x_b = \mathrm{diag}(i, -i)$, $K^{\theta_x} = \mathrm{diag}(z, \frac{1}{z}) \in \mathbb{C}^* \simeq SO(2, \mathbb{C})$

$K(\mathbb{R}) = SO(2), G(\mathbb{R}) = SU(1,1) = SL(2, \mathbb{R})$

$$K\backslash G/B = \{x_b = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, -x_b = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$$

$$x_b \to B = \begin{pmatrix} z & w \\ 0 & \frac{1}{z} \end{pmatrix}$$

$$-x_b = s_\alpha(x_b) \to B' = s_\alpha(B) = \begin{pmatrix} z & 0 \\ w & \frac{1}{z} \end{pmatrix}$$

$$u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, g = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \rightarrow$$

$$B'' = \begin{pmatrix} \cosh(z) & \sinh(z) \\ \sinh(z) & \cosh(z) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} w & w \\ -w & -w \end{pmatrix}$$

Key point:

$$H'' = \begin{pmatrix} \cosh(z) & \sinh(z) \\ \sinh(z) & \cosh(z) \end{pmatrix}$$

Split Cartan subgroup, $\theta = \theta_{x_b} = \mathrm{int}(\mathrm{diag}(i, -i))$ acts by -1

$$H''(\mathbb{R}) = \{ \begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix} \mid x \in \mathbb{R}\} \simeq \mathbb{R}^*$$

$$\boxed{(H'', \theta_{x_b}) \sim_g (H, \theta_u)}$$

LHS: how a human thinks of the split Cartan in SU(1,1) (fix $\theta = \theta_{x_b}$, vary the Cartan)

RHS: how Atlas thinks of the split Cartan: (fix H, vary $\theta = \theta_u$)

Moral of the Story

Always fixed fixed fixed: Cartan H $\subset$ Borel B

Fix $x_b, \theta = \mathrm{int}(x_b), K = G^\theta$

Vary $x \in \mathcal{X}, \theta_x$:

$\{(H', \theta)\}/K \leftrightarrow \{(H, \theta_x) \mid x \in \mathcal{X}\}$

$\{(B', \theta)\}/K \leftrightarrow \{(B, \theta_x) \mid x \in \mathcal{X}\}$

Also: $(\mathfrak{g}, K_x)$-modules *as x varies*

all equivalent to $(\mathfrak{g}, K_{x_b})$-modules

π a $(\mathfrak{g}, K_x)$ module, π' a $(\mathfrak{g}, K_{x'})$-module,
$\pi \simeq \pi'$ if $gxg^{-1} = x'$, $\pi^g \simeq \pi'$

More about KGB

Fix x_b, $\mathcal{X} = \mathcal{X}(x_b) \to K\backslash G/B \leftrightarrow \mathcal{X}$

$\texttt{KGB}{=}\mathcal{X} = \{x_0, \dots, x_n\}$

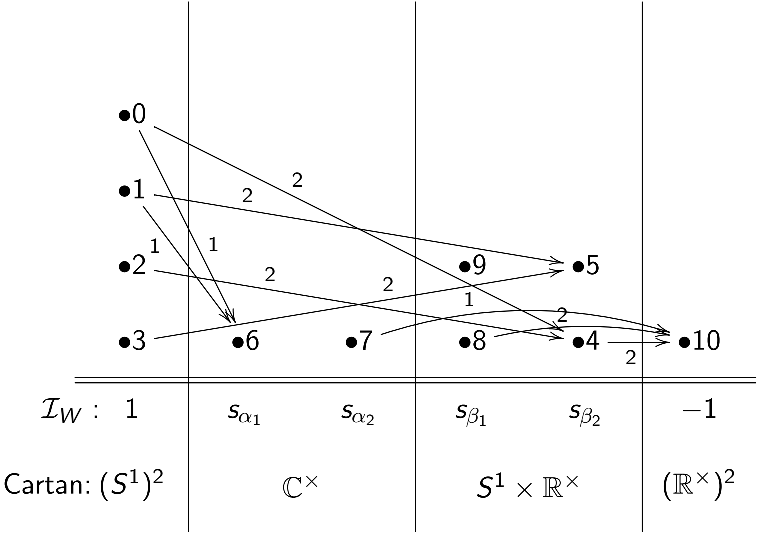
W acts naturally (conjugation) on $\mathcal{X}$

$\mathcal{X}/W \leftrightarrow$ conjugacy classes of Cartan subgroups

$\mathrm{Stab}_W(x) \simeq W(K, H) \simeq W(G(\mathbb{R}), H(\mathbb{R}))$

Map: $p : \mathcal{X} \to \mathcal{I}_W$ (involutions in W)

Example: KGB for $Sp(4,\mathbb{R})$



Discrete Series

Fix x_b

$\mathbb{F} := p^{-1}(\mathrm{Id}) = \{x \in \mathcal{X} \mid x \in H\}$ ("distinguished fiber")

Lemma:

$\mathcal{F} \leftrightarrow$ {Borel subgroups containing compact Cartan}$/K$

{Discrete series representations with fixed infinitesimal character}

$x = wx_b \to$ discrete series with Harish-Chandra parameter $w\rho$