Generalized Harish-Chandra Modules

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1 An introduction to generalized Harish-Chandra modules

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and $U\mathfrak{g}$ its universal enveloping algebra. Finite-dimensional representations of $\mathfrak{g}$ are well understood by now, but what can we say about infinite-dimensional modules? It is generally agreed that the theory of infinite-dimensional modules has revealed "wild" classification problems. That is, there is no systematic way of listing canonical forms of infinite-dimensional modules over a Lie algebra, even if they are simple, except for $\mathfrak{g} \simeq \mathfrak{sl}(2)$ (for this case see [B]). A standard text on the subject is J. Dixmier's book [D]. In general, we need to focus the problem further than just the study of simple modules in order to make any progress in understanding the theory of representations of semisimple Lie algebras.

One such focus is the theory of Harish-Chandra modules ([D,Ch.9],[Wa]).

Suppose σ is a nontrivial automorphism of order 2 of $\mathfrak{g}$. For example, if $\mathfrak{g} = \mathfrak{sl}(n)$, let $\sigma(T) = -T^t$ for all $T \in \mathfrak{g}$. Then $\mathfrak{g}^\sigma = \mathfrak{so}(n)$. In general we write $\mathfrak{k} = \mathfrak{g}^\sigma$ and call $\mathfrak{k}$ a *symmetric subalgebra* of $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathfrak{k})$ is called a *symmetric pair*. Any symmetric subalgebra $\mathfrak{k}$ is necessarily reductive in $\mathfrak{g}$, i.e. the adjoint representation of any symmetric subalgebra $\mathfrak{k} \subset \mathfrak{g}$ on $\mathfrak{g}$ is semisimple. For a given $\mathfrak{g}$, there are finitely many conjugacy classes of symmetric subalgebras, and there is always at least one. E. Cartan classified all symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ ([H],[Kn1]).

Definition 1.1 *A Harish-Chandra module for the pair* $(\mathfrak{g}, \mathfrak{k})$ *is a* $\mathfrak{g}$*-module* M *satisfying:*

1) M *is finitely generated over* $\mathfrak{g}$*.*

2) For all $v \in M$, $(U\mathfrak{k})v$ *is a finite-dimensional semisimple* $\mathfrak{k}$*-module.*

3) For any simple, finite-dimensional $\mathfrak{k}$*-module* V, $\dim \mathrm{Hom}_{\mathfrak{k}}(V, M) < \infty$*.*

Lemma 1.2 *If* M *is a Harish-Chandra module for* $(\mathfrak{g}, \mathfrak{k})$*, then we have a canonical decomposition of the restriction of* M *to* $\mathfrak{k}$*:*

$$M \cong \bigoplus_{V \in \mathrm{Rep}\mathfrak{k}} \mathrm{Hom}_{\mathfrak{k}}(V, M) \otimes_{\mathbb{C}} V,$$

where $\mathrm{Rep}\mathfrak{k}$ *is a complete set of representatives for the isomorphism classes of simple finite-dimensional* $\mathfrak{k}$*-modules.*

See Proposition 1.13 below for a more general statement.

The following is a classical result of Harish-Chandra.

Theorem 1.3 *([HC]; see [D, Ch.9]) Suppose* $\mathfrak{k}$ *is a symmetric subalgebra of* $\mathfrak{g}$*. Let* M *be a simple* $\mathfrak{g}$*-module which satisfies condition 2) of Definition 1.1. Then* M *satisfies condition 3) of Definition 1.1 .*

Let us attempt to put Harish-Chandra's theorem into some perspective. By $\mathfrak{l}$ we denote an arbitrary subalgebra of $\mathfrak{g}$.

Definition 1.4 *A* $(\mathfrak{g}, \mathfrak{l})$*-module is a* $\mathfrak{g}$*-module* M *such that for all* $v \in M$, $(U\mathfrak{l})v$ *is a finite-dimensional* $\mathfrak{l}$*-module (not necessarily semisimple over* $\mathfrak{l}$*).*

For example, let $\mathfrak{l} = \mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, that is, a maximal solvable subalgebra. Let E be a finite-dimensional $\mathfrak{b}$-module, and let $M = M(E) = U\mathfrak{g} \otimes_{U\mathfrak{b}} E$ be the $\mathfrak{g}$-module algebraically induced from E. Algebraic induction is introduced in [D, Ch.5].

When E is one dimensional, $M(E)$ is called a *Verma module* (although Harish-Chandra studied these objects long before Verma). For any finite-dimensional E, $M(E)$ is a $(\mathfrak{g}, \mathfrak{b})$-module; this follows from the following.

Lemma 1.5 *Let* E *be any finite-dimensional* $\mathfrak{l}$*-module. Then the induced module* $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ *is a* $(\mathfrak{g}, \mathfrak{l})$*-module.*

Proof Suppose $Y \in \mathfrak{l}, u \in U\mathfrak{g}$ and $e \in E$. Then in $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ we have

$$Y(u \otimes e) \quad = \quad Yu \otimes e = [Y, u] \otimes e + uY \otimes e = [Y, u] \otimes e + u \otimes Ye.$$

Write $U_{\mathrm{ad}\mathfrak{l}}$ for $U\mathfrak{g}$ regarded as an $\mathfrak{l}$-module via the adjoint action $\mathfrak{l}$. The above equation implies that we have an $\mathfrak{l}$-module surjection

$$U_{\mathrm{ad}\mathfrak{l}} \otimes_{\mathbb{C}} E \to U\mathfrak{g} \otimes_{U\mathfrak{l}} E.$$

The Poincare-Birkhoff-Witt (PBW) filtration of $U\mathfrak{g}$ yields a filtration of $U_{\mathrm{ad}\mathfrak{l}}$ by finite-dimensional $\mathfrak{l}$-submodules. Thus, $U_{\mathrm{ad}\mathfrak{l}} \otimes_{\mathbb{C}} E$ is locally finite as an $\mathfrak{l}$-module. By the above surjection, $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ is also locally finite as an $\mathfrak{l}$-module. □

Definition 1.6 *Let* M *be a countable-dimensional* $(\mathfrak{g}, \mathfrak{l})$*-module and let* V *be a finite-dimensional simple* $\mathfrak{l}$*-module.*

a) If W *is a finite-dimensional* $\mathfrak{l}$*-submodule of* M*, let* $[W : V]$ *be the multiplicity of* V *as a Jordan-Hölder factor of* W*.*

b) Let $[M : V]$ *be the supremum of* $[W : V]$ *as* W *runs over all finite-dimensional* $\mathfrak{l}$*-submodules of* M*. We call* $[M : V]$ *the multiplicity of* V *in* M*. We have* $[M : V] \in \mathbb{N} \cup \{\omega\}$*, where* ω *stands for the countable infinite cardinal.*

Lemma 1.7 *If* E *is a finite-dimensional* $\mathfrak{l}$*-module and* V *is a simple finite-dimensional* $\mathfrak{l}$*-module, then*

$$[U\mathfrak{g} \otimes_{U\mathfrak{l}} E : V] = [S(\mathfrak{g}/\mathfrak{l}) \otimes_{\mathbb{C}} E : V].$$

Proof The PBW filtration of $U\mathfrak{g}$ is ad$\mathfrak{l}$-stable. This filtration yields a filtration of $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ whose associated graded module is $S(\mathfrak{g}/\mathfrak{l}) \otimes_{\mathbb{C}} E$ (here and bellow $S(\ )$ stands for symmetric algebra). □

Definition 1.8 *a) A* $(\mathfrak{g}, \mathfrak{l})$*-module* M *has finite type over* $\mathfrak{l}$ *if* $[M : V] < \infty$ *for any simple finite-dimensional* $\mathfrak{l}$*-module* V*.*

b) A generalized Harish-Chandra module is a $\mathfrak{g}$*-module which is of finite type for some* $\mathfrak{l}$ *in* $\mathfrak{g}$*, not necessarily specified in advance.(See [PZ1], [PSZ].)*

Example 1.9

Suppose $\mathfrak{b}$ is a Borel subalgebra and E is a finite-dimensional $\mathfrak{b}$-module. Then $U\mathfrak{g} \otimes_{U\mathfrak{b}} E$ has finite type over $\mathfrak{b}$. Indeed let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{b}$. As an $\mathfrak{h}$-module, $S(\mathfrak{g}/\mathfrak{b})$ is semisimple with finite multiplicities. Hence $S(\mathfrak{g}/\mathfrak{b})$ has finite multiplicities as a $\mathfrak{b}$-module. Now apply Lemma 1.7.

Proposition 1.10 *Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, and M be a simple $(\mathfrak{g}, \mathfrak{b})$-module.*

a) M is semisimple over $\mathfrak{h}$ with finite multiplicities.

b) There exists a unique one-dimensional $\mathfrak{b}$-module E such that M is a quotient of $M(E) := U\mathfrak{g} \otimes_{U\mathfrak{b}} E$.

c) $M(E)$ has a unique simple quotient.

Proof Part b): Let E be a one-dimensional $\mathfrak{b}$-submodule of M (E exists by Lie's Theorem). The embedding of E into M yields a surjective homomorphism of $M(E)$ onto M.

Part a): Follows from the same statement for $M(E)$.

Part c): See [D, Ch.7]. □

Recall that $\mathfrak{l}$ is an arbitrary subalgebra of $\mathfrak{g}$ and let E be a finite-dimensional $\mathfrak{l}$-module. In general, $S(\mathfrak{g}/\mathfrak{l})$ has infinite multiplicities as an $\mathfrak{l}$-module. Hence, $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ has infinite multiplicities as an $\mathfrak{l}$-module. Now suppose M is a simple $(\mathfrak{g}, \mathfrak{l})$-module. By Schur's Lemma [D, Ch. 2, Sec. 6], the center $Z_{U\mathfrak{g}}$ of the enveloping algebra $U\mathfrak{g}$ will act via scalars on M. Let $\theta_M : Z_{U\mathfrak{g}} \to \mathbb{C}$ be the corresponding central character of M. Let E be a non-zero finite-dimensional $\mathfrak{l}$-submodule of M. Then, M is a quotient of the $(\mathfrak{g}, \mathfrak{l})$-module

$$P(E, \theta_M) = (U\mathfrak{g} \otimes_{U\mathfrak{l}} E) \otimes_{Z_{U\mathfrak{g}}} (Z_{U\mathfrak{g}}/\mathrm{Ker}\theta_M).$$

In general $P(E, \theta_M)$ has infinite multiplicities as an $\mathfrak{l}$-module.

The following is a crucial fact leading to the proof of Harish-Chandra's Theorem (Theorem 1.3).

Proposition 1.11 *If $\mathfrak{k}$ is a symmetric subalgebra of $\mathfrak{g}$, E is a simple finite-dimensional $\mathfrak{k}$-module and θ is a homomorphism from $Z_{U\mathfrak{g}}$ to $\mathbb{C}$, then $P(E, \theta)$ has finite type over $\mathfrak{k}$.*

Proof See [D, Ch.9], [W]. □

Proposition 1.11 implies Theorem 1.3 as, if M is a simple $(\mathfrak{g}, \mathfrak{k})$-module with central character θ_M and simple $\mathfrak{k}$-submodule E, then M is a quotient of $P(E, \theta_M)$.

Remark. For general E and θ, $P(E, \theta)$ could vanish.

For later use we state the following.

Lemma 1.12 *If $\mathfrak{l}$ is reductive in $\mathfrak{g}$ and M is a simple $(\mathfrak{g}, \mathfrak{l})$-module, then M is semisimple over $\mathfrak{l}$.*

Proof Let E be a simple $\mathfrak{l}$-submodule of M. We have a canonical homomorphism of $U\mathfrak{g} \otimes_{U\mathfrak{l}} E \to M$ given by $u \otimes e \mapsto ue$. Since M is simple, this homomorphism is surjective. In turn, the homomorphism $U_{\mathrm{ad}\mathfrak{l}} \otimes_{\mathbb{C}} E \to U\mathfrak{g} \otimes_{U\mathfrak{l}} E$, given by $u \otimes e \mapsto u \otimes e \in U\mathfrak{g} \otimes_{U\mathfrak{l}} E$, is surjective. Since $\mathfrak{l}$ is reductive in $\mathfrak{g}$, $U_{\mathrm{ad}\mathfrak{l}}$ is a semisimple $\mathfrak{l}$-module; since the $\mathfrak{l}$-module E is simple, $U_{\mathrm{ad}\mathfrak{l}} \otimes_{\mathbb{C}} E$ is a semisimple $\mathfrak{l}$-module. Hence, $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ and M itself are semisimple over $\mathfrak{l}$. □

Proposition 1.13 *If $\mathfrak{l}$ is reductive in $\mathfrak{g}$ and M is a simple $(\mathfrak{g}, \mathfrak{l})$-module, then as an $\mathfrak{l}$-module, M is canonically isomorphic to*

$$\bigoplus_{V \in \mathrm{Rep}\mathfrak{l}} \mathrm{Hom}_{\mathfrak{l}}(V, M) \otimes V.$$

In particular, M has finite type over $\mathfrak{l}$ if and only if for every $V \in \mathrm{Rep}\mathfrak{l}$, $\mathrm{Hom}_{\mathfrak{l}}(V, M)$ is finite dimensional. In general, $[M : V] = \dim \mathrm{Hom}_{\mathfrak{l}}(V, M)$.

Proof See [Kn2]. □

Consider now the case $\mathfrak{l} = \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, the maximal nilpotent subalgebra of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. For any finite-dimensional $\mathfrak{n}$-module F, the algebraically induced module $U\mathfrak{g} \otimes_{U\mathfrak{n}} F$ is a $(\mathfrak{g}, \mathfrak{n})$-module, and by tensoring over $Z_{U\mathfrak{g}}$ with a finite-dimensional representation V of $Z_{U\mathfrak{g}}$,

$$(U\mathfrak{g} \otimes_{U\mathfrak{n}} F) \otimes_{Z_{U\mathfrak{g}}} V, \tag{1.1}$$

we again obtain a $(\mathfrak{g}, \mathfrak{n})$-module. Since $\mathfrak{n}$ is not symmetric, this does not guarantee finite multiplicities.

Definition 1.14 *A one-dimensional $\mathfrak{n}$-module F is generic if for each simple root α of $\mathfrak{b}$ in $\mathfrak{g}$, $\mathfrak{g}_\alpha$ acts nontrivially on F (note that $\mathfrak{g}_\alpha \subset \mathfrak{n}$).*

In fact, we have the following well-known result.

Theorem 1.15 *([Ko]) If F is one-dimensional and generic, and V is one-dimensional, then $(U\mathfrak{g} \otimes_{U\mathfrak{n}} F) \otimes_{Z_{U\mathfrak{g}}} V$ is simple, but F occurs with infinite multiplicity.*

The reader is now urged to compare Theorem 1.15 to Proposition 1.11. In fact, one can show that the $\mathfrak{g}$-module $(U\mathfrak{g} \otimes_{U\mathfrak{n}} F) \otimes_{Z_{U\mathfrak{g}}} V$ is not a generalized Harish-Chandra module.

Let $\mathfrak{l} = \mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and M a simple $(\mathfrak{g}, \mathfrak{h})$-module. By Proposition 1.13, M is a direct sum of weight spaces (joint eigenspaces) of $\mathfrak{h}$. Each weight space corresponds to a linear functional on $\mathfrak{h}$, so we may write

$$M = \bigoplus_{\lambda \in \mathfrak{h}*} M(\lambda). \tag{1.2}$$

Although the dual space $\mathfrak{h}^*$ is uncountable, this sum is in fact supported on a countable set of weights, which we denote $\mathrm{supp}_{\mathfrak{h}} M$. In what follows we call any $\mathfrak{g}$-module satisfying (2) an $\mathfrak{h}$-*weight module* or simply a *weight module.*

The decomposition (2) should in principle allow us to understand weight modules better since the root decomposition of $\mathfrak{g}$ respects this decomposition of any $(\mathfrak{g}, \mathfrak{k})$-module M. That is, if α is a root of $\mathfrak{h}$ in $\mathfrak{g}$, X_α a nonzero root vector, then $X_\alpha M(\lambda) \subseteq M(\lambda + \alpha)$. However, there is no classification of simple weight modules for the pair $(\mathfrak{g}, \mathfrak{h})$ unless $\mathfrak{g} \simeq \mathfrak{sl}(2)$. The classification problem appears to be wild already for $\mathfrak{g} \simeq \mathfrak{sl}(3)$, although for $\mathfrak{sl}(2)$ it was solved in the 1940's, and in fact is given as an exercise in [D].

Consider the case $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g} = \mathfrak{sl}(n)$ with $\mathfrak{k}$ the symmetric subalgebra consisting of elements of the type below:

$$\begin{pmatrix} * & ... & * & 0 \\ \vdots & & \vdots & \vdots \\ * & ... & * & 0 \\ 0 & ... & 0 & * \end{pmatrix} \simeq \mathfrak{sl}(n-1) \oplus \mathbb{C} \tag{1.3}$$

(here $\mathfrak{h}$ is the subalgebra of traceless diagonal matrices).

We have the following known types of simple $(\mathfrak{g}, \mathfrak{k})$-modules (due to H. Kraljević [Kra]).

1. Those which have infinite $\mathfrak{h}$-multiplicities.

2. Modules which are $(\mathfrak{g}, \mathfrak{b})$-modules for some Borel containing $\mathfrak{h}$. Such modules have finite $\mathfrak{h}$-multiplicities.

Kraljević gives an explicit construction of a complete set of representatives for the isomorphism classes of simple $(\mathfrak{g}, \mathfrak{k})$-modules. The completeness of this set is implied by the Harish-Chandra subquotient theorem, see [D, Ch. 9].

For $\mathfrak{g} = \mathfrak{sl}(n)$, there exist simple $(\mathfrak{g}, \mathfrak{h})$-modules which are not $(\mathfrak{g}, \mathfrak{b})$-modules for any Borel $\mathfrak{b}$ containing $\mathfrak{h}$, but which have finite $\mathfrak{h}$-multiplicities. Britten and Lemire showed in 1982 that we can construct a module M such that $\mathrm{supp}_\mathfrak{h} M = \nu + \Lambda$, where Λ is the root lattice of $\mathfrak{h}$, and ν is a completely non-integral weight of $\mathfrak{h}$. Moreover, if $\lambda \in \mathrm{supp}_\mathfrak{h} M$, then $\dim M(\lambda) = 1$. (See [BL], [BBL].)

Here is an explicit construction of Britten-Lemire modules. Let $x_1, x_2, ..., x_n$ be coordinates for $\mathbb{C}^n$. If $\lambda \in \mathfrak{h}^*$, write $\lambda = (\lambda_1,\ \lambda_2, ...,\ \lambda_n)$, with $\lambda_1 + \lambda_2 + ... + \lambda_n = 0$. Let x^λ denote the formal monomial $x_1^{\lambda_1} x_2^{\lambda_2} ... x_n^{\lambda_n}$, and define $\frac{\partial}{\partial x_i} x^\lambda = \lambda_i x^{\lambda - (\delta_{i1}, \delta_{i2}, ..., \delta_{in})}$ where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. Let F be the $\mathbb{C}$ vector space with basis $\{x^\lambda | \lambda \in \mathfrak{h}^*\}$. Then F is a module over the Lie algebra $\tilde{\mathfrak{g}}$ spanned by the vector fields $x_i \frac{\partial}{\partial x_j}$. Identify $\mathfrak{g} = \mathfrak{sl}(n)$ with the subalgebra $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ of $\tilde{\mathfrak{g}}$.
Then $\mathfrak{h} = \mathrm{span}\{x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\}$, and the λ-weight space of F is precisely $F(\lambda) = \mathbb{C}x^\lambda$, for each $\lambda \in \mathfrak{h}^*$. Fix a weight $\nu \in \mathfrak{h}^*$ and let $M_\nu = (U\mathfrak{g}) \cdot x^\nu \subseteq F$. Assume $\nu_i \notin \mathbb{Z}$ for $i = 1, 2, ..., n$. Then M_ν is a simple $(\mathfrak{g}, \mathfrak{h})$-module with $\mathrm{supp}_\mathfrak{h} M_\nu = \nu + \Lambda$.

For $\mathfrak{g}$ an arbitrary finite-dimensional Lie algebra, M an arbitrary $\mathfrak{g}$-module, define

$$\mathfrak{g}[M] = \{Y \in \mathfrak{g} |\ \mathbb{C}Y \text{ acts locally finitely in } M\}. \tag{1.4}$$

Theorem 1.16 [1] *(S. Fernando [F], V. Kac [K]) The subset $\mathfrak{g}[M]$ is a subalgebra of $\mathfrak{g}$, called the Fernando-Kac subalgebra associated to M.*

Note that even for a simple $\mathfrak{g}$-module M, we may have $\mathfrak{g}[M] = 0$: such an example for $\mathfrak{g} = \mathfrak{sl}(2)$ was found by D. Arnal and G. Pinczon [AP].

Any $\mathfrak{g}$-module M is a $(\mathfrak{g}, \mathfrak{g}[M])$-module. Moreover, M is a generalized Harish-Chandra module if and only if M is of finite type over $\mathfrak{g}[M]$. (See Definition 1.8.)

At this point, we have a theory moving in two directions. To a subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, we associate the category of $(\mathfrak{g}, \mathfrak{l})$-modules. We can also associate to $\mathfrak{l}$ the subcategory of $(\mathfrak{g}, \mathfrak{l})$-modules of finite type. On the other hand, the Fernando-Kac construction allows us to identify a subalgebra $\mathfrak{g}[M]$ of $\mathfrak{g}$ for every module M. That is, if $\mathcal{C}(\mathfrak{g})$ is the category of all $\mathfrak{g}$-modules,

$$\mathfrak{g}[-] : \mathcal{C}(\mathfrak{g}) \to Sub(\mathfrak{g})$$

is a map from the class of objects of $\mathcal{C}(\mathfrak{g})$ to the set $Sub(\mathfrak{g})$ of subalgebras of $\mathfrak{g}$.

Theorem 1.17 *([PS]) Let $\mathfrak{g}$ be reductive. Every subalgebra between $\mathfrak{h}$ and $\mathfrak{g}$, i.e. every root subalgebra of $\mathfrak{g}$, arises as the Fernando-Kac subalgebra of some simple weight module of $\mathfrak{g}$.*

As we will see below for $\mathfrak{g} = \mathfrak{sl}(3)$, for certain root subalgebras $\mathfrak{l} \supset \mathfrak{h}$, the equality $\mathfrak{g}[M] = \mathfrak{l}$ for a simple $\mathfrak{g}$-module M implies that M has infinite type over $\mathfrak{h}$.

Consider now the case when $\mathfrak{g}$ is simple and $\mathfrak{k}$ is a proper symmetric subalgebra of $\mathfrak{g}$. There are two cases.

1. The center of $\mathfrak{k}$ is trivial.

   In this case $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$. If M is a simple infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module, $\mathfrak{g}[M] = \mathfrak{k}$.

2. The center of $\mathfrak{k}$ is nontrivial. (Example: $\mathfrak{g} = \mathfrak{sl}(n), \mathfrak{k} = \mathfrak{sl}(n-1) \oplus \mathbb{C}$.)

   In this case $Z\mathfrak{k} \cong \mathbb{C}$ and $\mathfrak{k}$ is the reductive part of two opposite maximal parabolic subalgebras $\mathfrak{p}_+$ and $\mathfrak{p}_-$ in $\mathfrak{g}$. Moreover, the only subalgebras lying between $\mathfrak{k}$ and $\mathfrak{g}$ are $\mathfrak{k}, \mathfrak{p}_+, \mathfrak{p}_-$ and $\mathfrak{g}$. If M is a simple infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module, $\mathfrak{g}[M]$ can be any of the three subalgebras $\mathfrak{k}, \mathfrak{p}_+$ or $\mathfrak{p}_-$. These facts are a consequence of the theory of the Fernando-Kac subalgebra plus early work of Harish-Chandra ([D], Ch. 9).

The case of $\mathfrak{l}$ simple, $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}$, $\mathfrak{k} = \mathfrak{l}$ embedded diagonally into $\mathfrak{g}$ is also interesting, since $\mathfrak{l}$ is maximal in $\mathfrak{g}$.

For $\mathfrak{g} = \mathfrak{sl}(3)$, $\mathfrak{h}$ diagonal matrices, the simple $(\mathfrak{g}, \mathfrak{h})$-modules considered so far have the following Fernando-Kac subalgebras.

1. If M is finite dimensional, then $\mathfrak{g}[M] = \mathfrak{g}$ (i.e. every $Y \in \mathfrak{g}$ acts locally finitely on M).

[1] We thank A. Joseph for pointing out that Theorem 1.16 follows also from an earlier result of B. Kostant reproduced in [GQS].

2. If M is a simple infinite-dimensional $(\mathfrak{g}, \mathfrak{b})$-module for a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$, then $\mathfrak{b} \subseteq \mathfrak{g}[M]$, so $\mathfrak{g}[M]$ is a parabolic subalgebra of $\mathfrak{g}$.

3. For M a Britten-Lemire module, $\mathfrak{g}[M] = \mathfrak{h}$.

4. Let $\mathfrak{k} = \mathfrak{sl}(2) \oplus \mathbb{C}$, i.e. $\mathfrak{k} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$. For M a simple $(\mathfrak{g}, \mathfrak{k})$-module which is not a $(\mathfrak{g}, \mathfrak{b})$-module for any Borel subalgebra $\mathfrak{b}$, we have $\mathfrak{g}[M] = \mathfrak{k}$.

These four classes are distinguished by $\mathfrak{g}[M]$. There are many additional modules, and it is interesting to ask which subalgebras of $\mathfrak{g}$ can occur as $\mathfrak{g}[M]$.

5. Let $\mathfrak{l} = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$. By [F], any simple M with $\mathfrak{g}[M] = \mathfrak{l}$ has finite type over $\mathfrak{l}$.

6. Let $\mathfrak{l} = \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$. By [F], any simple M with $\mathfrak{g}[M] = \mathfrak{l}$ has infinite type over $\mathfrak{l}$.

The cases 1.-6. exhaust all root subalgebras of $\mathfrak{sl}(3)$ up to conjugacy. Case 3, i.e. when $\mathfrak{l} = \mathfrak{h}$, is the only case when simple $(\mathfrak{g}, \mathfrak{l})$-modules M with $\mathfrak{g}[M] = \mathfrak{h}$ can have both finite or infinite type. In particular, there exist simple $\mathfrak{g}$-modules M with $\mathfrak{g}[M] = \mathfrak{h}$ which have infinite type over $\mathfrak{h}$. See [Fu1], [Fu2], [Fu3], and [PS].

The following theorem gives a general characterization of the subalgebra $\mathfrak{g}[M]$ corresponding to a generalized Harish-Chandra module M. By $\ni\!\!\!\!\supset$ we denote the semi-direct sum of Lie algebras: the round part of the sign points towards the ideal.

Theorem 1.18 *([PSZ]) Assume $\mathfrak{g}$ is semisimple. Suppose M is a simple generalized Harish-Chandra module.*

(a) $\mathfrak{g}[M]$ is self-normalizing, hence $\mathfrak{g}[M]$ is algebraic. That is, $\mathfrak{g}[M]$ is the Lie algebra of a subgroup of $G = Aut(\mathfrak{g})^\circ$ $((\ )^\circ$ indicates the connected component of the identity).

(b) Let $\mathfrak{k} \subseteq \mathfrak{g}[M]$ be maximal among subalgebras of $\mathfrak{g}[M]$ that are reductive as subalgebras of $\mathfrak{g}$. Let $\mathfrak{g}[M]_{nil}$ be the maximum ad-nilpotent ideal in $\mathfrak{g}[M]$. Then, $\mathfrak{g}[M] = \mathfrak{k} \ni\!\!\!\!\supset \mathfrak{g}[M]_{nil}$. Although $\mathfrak{k}$ is only unique up to conjugation, write $\mathfrak{k} = \mathfrak{g}[M]_{red}$.

(c) $M \in \mathcal{A}(\mathfrak{g}, \mathfrak{g}[M]_{red})$. Moreover, $\mathfrak{g}[M]_{red}$ acts semi-simply on M.

(d) The center $Z(\mathfrak{g}[M]_{red})$ of $\mathfrak{g}[M]_{red}$ is equal to the centralizer $C_{\mathfrak{g}}(\mathfrak{g}[M]_{red})$ of $\mathfrak{g}[M]_{red}$ in $\mathfrak{g}$.

If we are just interested in simple generalized Harish-Chandra modules, we can give a more transparent characterization of them. A simple $\mathfrak{g}$-module M is a generalized Harish-Chandra module if M is a direct sum with finite multiplicities of finite-dimensional simple $\mathfrak{g}[M]_{red}$-modules.

For M as in Theorem 1.18 , there is a canonical isomorphism

$$M \xleftarrow{\sim} \bigoplus_{F \in \mathrm{Rep}(\mathfrak{g}[M]_{red})} \mathrm{Hom}_{\mathfrak{g}[M]_{red}}(F, M) \otimes_{\mathbb{C}} F. \tag{1.5}$$

Choose $\mathfrak{t}$ a Cartan subalgebra in $\mathfrak{k} = \mathfrak{g}[M]_{\mathrm{red}}$, and fix a root order for $\mathfrak{t}$ in $\mathfrak{k}$. Then, Rep$\mathfrak{k}$ bijects to the dominant integral weights $\Lambda_d \subset \mathfrak{t}^*$. Therefore, we can rewrite M as

$$M \xleftarrow{\sim} \bigoplus_{\lambda \in \Lambda_d} \mathrm{Hom}_{\mathfrak{k}}(F_\lambda, M) \otimes_{\mathbb{C}} F_\lambda \tag{1.6}$$

with F_λ the simple finite-dimensional module of highest weight λ. Let $M[\lambda]$ be the image of $\mathrm{Hom}_{\mathfrak{k}}(F_\lambda, M) \otimes_{\mathbb{C}} F_\lambda$ under the isomorphism in (1.6). We call $M[\lambda]$ the *F_λ-isotypic $\mathfrak{k}$-submodule of M.*

Fix $\mathfrak{k}$ a reductive in $\mathfrak{g}$ subalgebra. Assume (as in Theorem 1.18 (d)) that $C_{\mathfrak{g}}(\mathfrak{k}) = Z(\mathfrak{k})$. How might one construct simple generalized Harish-Chandra modules M such that $\mathfrak{k} = \mathfrak{g}[M]_{red}$?

Theorem 1.19 *([PSZ]) At least one simple M exists satisfying the above.*

The proof involves quite a bit of algebraic geometry. Basically, we can employ the natural action of K on certain partial flag varieties for G (here G and K are the connected algebraic groups corresponding to $\mathfrak{g}$ and $\mathfrak{k}$), then view M as global sections of a sheaf on this variety.

2 An introduction to the Zuckerman functor

In the following construction of $(\mathfrak{g}, \mathfrak{k})$-modules via derived functors, we will have three primary goals:

1. Systematically construct generalized Harish-Chandra modules for this pair.
2. Give a classification of a natural class of simple generalized Harish-Chandra modules.
3. Calculate $\mathfrak{g}[M]$ for a class of modules M constructed via derived functors.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, with $\mathfrak{m} \subsetneq \mathfrak{k}$ a pair of reductive in $\mathfrak{g}$ subalgebras. We have the category $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ of $\mathfrak{g}$-modules which are locally finite and completely reducible over $\mathfrak{k}$, and likewise the category $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$. Both categories are closed under taking submodules, quotients, arbitrary direct sums, tensor products over $\mathbb{C}$, and $\Gamma'\mathrm{Hom}_{\mathbb{C}}(-,-)$. (See the discussion below for the definition of Γ'.)

Note that $A \in \mathcal{C}(\mathfrak{g}, \mathfrak{k})$ implies $A \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$. Therefore, we can define a forgetful functor $For : \mathcal{C}(\mathfrak{g}, \mathfrak{k}) \to \mathcal{C}(\mathfrak{g}, \mathfrak{m})$. For general homological algebra reasons, this functor has a right adjoint $\Gamma : \mathcal{C}(\mathfrak{g},\mathfrak{m}) \to \mathcal{C}(\mathfrak{g},\mathfrak{k})$ unique up to isomorphism. Recall that Γ is a right adjoint to For means there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}(\mathfrak{g},\mathfrak{m})}(For(A), V) \simeq \mathrm{Hom}_{\mathcal{C}(\mathfrak{g},\mathfrak{k})}(A, \Gamma V). \tag{2.1}$$

An explicit construction for Γ is as follows. For $V \in \mathcal{C}(\mathfrak{g},\mathfrak{m})$, let ΓV be the sum in V of all cyclic $\mathfrak{g}$-submodules B such that $B \in \mathcal{C}(\mathfrak{g},\mathfrak{k})$.

Proposition 2.1 *As a set, ΓV is the set of $v \in V$ such that $(U\mathfrak{k})v$ is finite dimensional and semisimple as a $\mathfrak{k}$-module.*

Proof Suppose that, for some $v \in V$, $E = (U\mathfrak{k})v$ is finite dimensional and semisimple over $\mathfrak{k}$. Then the cyclic submodule $B = (U\mathfrak{g})v$ is a quotient of $U\mathfrak{g} \otimes_{U\mathfrak{k}} E$, which is an object in $\mathcal{C}(\mathfrak{g},\mathfrak{k})$. Hence, $B \in \mathcal{C}(\mathfrak{g},\mathfrak{k})$.

Conversely, if $B = (U\mathfrak{g})v$ is in $\mathcal{C}(\mathfrak{g},\mathfrak{k})$, then $(U\mathfrak{k})v$ is finite dimensional and semisimple over $\mathfrak{k}$. If $v_1, v_2, \ldots, v_n$ are elements of V such that for each i, $B_i = (U\mathfrak{g})v_i$ is in $\mathcal{C}(\mathfrak{g},\mathfrak{k})$, then for any element w in $B_1 + B_2 + \ldots + B_n$, $(U\mathfrak{k})w$ is finite dimensional and semisimple over $\mathfrak{k}$. □

A more conceptual way to think of ΓV is as the largest $(\mathfrak{g},\mathfrak{k})$-module in V.

The functor Γ is left exact, but happens to be not right exact. To see a simple example of failure of right exactness, consider the exact sequence

$$0 \to (U\mathfrak{g})\mathfrak{g} \to U\mathfrak{g} \to \mathbb{C} \to 0. \tag{2.2}$$

Set $\mathfrak{k} = \mathfrak{g}, \mathfrak{m} = 0$. Then, Γ takes $U\mathfrak{g}$ and the augmentation ideal $(U\mathfrak{g})\mathfrak{g}$ to 0, but $\Gamma\mathbb{C} = \mathbb{C}$.

Once we know Γ is not exact, we should have a Pavlovian response and try to define corresponding derived functors. Existence of these functors is contingent on the existence of enough injectives in the category $\mathcal{C}(\mathfrak{g},\mathfrak{m})$.

Lemma 2.2 *$\mathcal{C}(\mathfrak{g},\mathfrak{m})$ has enough injectives.*

Proof Take any $V \in \mathcal{C}(\mathfrak{g},\mathfrak{m})$. Embed V in an injective $\mathfrak{g}$-module X. Write $\Gamma' : \mathcal{C}(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g},\mathfrak{m})$ for the adjoint of the inclusion $\mathcal{C}(\mathfrak{g},\mathfrak{m}) \subset \mathcal{C}(\mathfrak{g})$. Here, $\Gamma' V = V$ and so $V \to \Gamma' X$ is an embedding of V to an injective object in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$. That $\Gamma' X$ is injective is another standard homological algebra fact which follows from Γ' being right adjoint to For. By repeating these steps, we can prolong the embedding $V \to \Gamma' X$ to an injective resolution

$$0 \to V \to I^0 \to I^1 \to I^2 \to \ldots. \tag{2.3}$$

in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$. □

Let $I^\bullet$ denote the resolution $0 \to I^0 \to I^1 \to \ldots$. Now, we can define

$$R^p\Gamma(V) := \mathrm{H}^p(\Gamma I^\bullet).$$

The maps in $0 \to \Gamma I^0 \to \Gamma I^1 \to \ldots$ are maps of $(\mathfrak{g},\mathfrak{k})$-modules, therefore $\mathrm{H}^*(\Gamma I^\bullet)$ is a $\mathbb{Z}$-graded $(\mathfrak{g},\mathfrak{k})$-module.

We may introduce more systematic notation to identify the categories in which we are working:

$$\Gamma^{\mathfrak{g},\mathfrak{m}}_{\mathfrak{g},\mathfrak{k}} : \mathcal{C}(\mathfrak{g},\mathfrak{m}) \to \mathcal{C}(\mathfrak{g},\mathfrak{k}). \tag{2.4}$$

Likewise we can write the derived functors $R\Gamma^{\mathfrak{g},\mathfrak{m}}_{\mathfrak{g},\mathfrak{k}}$. In the literature, $R^*\Gamma^{\mathfrak{g},\mathfrak{m}}_{\mathfrak{g},\mathfrak{k}}$ have been referred to as the Zuckerman functors, see [V]. These derived functors depend on a choice of injective resolution, but they are well-defined up to isomorphism.

In $\mathcal{C}(\mathfrak{g},\mathfrak{m})$ we can construct a functorial resolution, by using the relative Koszul complex. Recall that $K_\bullet(\mathfrak{g},\mathfrak{m})$ is given by $K_i(\mathfrak{g},\mathfrak{m}) = U\mathfrak{g} \otimes_{U\mathfrak{m}} \Lambda^i(\mathfrak{g}/\mathfrak{m})$ with Koszul differential $\partial_i : K_i(\mathfrak{g},\mathfrak{m}) \to K_{i-1}(\mathfrak{g},\mathfrak{m})$, see [BW]. The complex $K_\bullet(\mathfrak{g},\mathfrak{m})$ is acyclic and yields a resolution of $\mathbb{C}$. If V is a $(\mathfrak{g},\mathfrak{m})$-module, let $I^i(V) = \Gamma'\mathrm{Hom}_{\mathbb{C}}(K_i(\mathfrak{g},\mathfrak{m}),V)$. Then for every i, $I^i(V)$ is an injective object in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$. Moreover, $I^\bullet(V)$ is a resolution of V. Finally, $V \rightsquigarrow I^\bullet(V)$ is an exact functor.

As an application, we can write

$$R^*\Gamma^{\mathfrak{g},\mathfrak{m}}_{\mathfrak{g},\mathfrak{k}} V \cong H^*(\Gamma^{\mathfrak{g},\mathfrak{m}}_{\mathfrak{g},\mathfrak{k}}\mathrm{Hom}_{\mathbb{C}}(K_\bullet(\mathfrak{g},\mathfrak{m}),V)).$$

This formula makes clear the dependence of $R^*\Gamma^{\mathfrak{g},\mathfrak{m}}_{\mathfrak{g},\mathfrak{k}}$ on the triple $(\mathfrak{g},\mathfrak{k},\mathfrak{m})$. Note that we have nowhere used the assumption that $\mathfrak{g}$ is finite dimensional. However, we have used in an essential way that $\mathfrak{m}$ and $\mathfrak{k}$ are finite dimensional and act semisimply on $\mathfrak{g}$ via the adjoint representation. See [PZ4] for an application to infinite-dimensional $\mathfrak{g}$.

Example 2.3

Let $\mathfrak{g}$ be semisimple, $\mathfrak{m} = \mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{k}$ be any reductive in $\mathfrak{g}$ subalgebra containing $\mathfrak{h}$. Let $V \in \mathcal{C}(\mathfrak{g},\mathfrak{h})$. The module V is a weight module. If V has finite $\mathfrak{h}$-multiplicities, then $R^*\Gamma^{\mathfrak{g},\mathfrak{h}}_{\mathfrak{g},\mathfrak{k}} V$ has finite $\mathfrak{k}$-multiplicities. (See Theorem 2.4 below).

For a general triple $\mathfrak{m} \subset \mathfrak{k} \subset \mathfrak{g}$, let $\mathcal{A}(\mathfrak{g},\mathfrak{m})$ denote the category of $(\mathfrak{g},\mathfrak{m})$-modules which are semisimple over $\mathfrak{m}$ and have finite multiplicities.

Theorem 2.4 *Set* $\Gamma = \Gamma^{\mathfrak{g},\mathfrak{m}}_{\mathfrak{g},\mathfrak{k}}$ *and let* $M \in \mathcal{A}(\mathfrak{g},\mathfrak{m})$.

a) For all i, $R^i\Gamma M \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$.

b) If $i > \dim \mathfrak{k}/\mathfrak{m}$, *then* $R^i\Gamma M = 0$.

c) $\bigoplus_{i\in\mathbb{N}} R^i\Gamma M \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$.

Before we give the proof of Theorem 2.4 we introduce a generalization of our setup. The following more general assumptions are in effect up to Corollary 2.8 included. Assume $\mathfrak{g}$ is finite dimensional, but no longer assume that $\mathfrak{k}$ is reductive in $\mathfrak{g}$. Let $\mathfrak{k}_r \subset \mathfrak{k}$ be maximal among subalgebras of $\mathfrak{k}$ that are reductive in $\mathfrak{g}$, and let $\mathfrak{m}$ be reductive in $\mathfrak{k}_r$. Denote by $\mathcal{C}(\mathfrak{g},\mathfrak{k},\mathfrak{k}_r)$ the full subcategory of $\mathfrak{g}$-modules M such that M is a $(\mathfrak{g},\mathfrak{k})$-module which is semisimple over $\mathfrak{k}_r$.

Example 2.5

Let $\mathfrak{k} = \mathfrak{b}$, a Borel subalgebra of $\mathfrak{g}$. Choose $\mathfrak{k}_r = \mathfrak{h}$, a Cartan subalgebra of $\mathfrak{g}$ that lies in $\mathfrak{b}$. The subcategory $\mathcal{O}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ of finitely generated modules in $\mathcal{C}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ was introduced by Bernstein-Gelfand-Gelfand [BGG].

As before, $\mathcal{C}(\mathfrak{g}, \mathfrak{k}, \mathfrak{k}_r)$ is a full subcategory of $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$ and we can construct a right adjoint Γ to this inclusion of categories. Likewise, we can study the right derived functors of Γ. It is interesting to understand when $R^*\Gamma M$ has finite type over $\mathfrak{k}$. A simpler question is the following.

Problem: Suppose V is a simple finite-dimensional $\mathfrak{k}_r$-module. Under what conditions on V and the data above will $\dim \mathrm{Hom}_{\mathfrak{k}_r}(V, R^*\Gamma M)$ be finite?

The question of when $R^*\Gamma M$ has finite type over $\mathfrak{k}$ is related to the question of when $\dim \mathrm{Hom}_{\mathfrak{k}}(Z, R^*\Gamma M)$ is finite for finite-dimensional (not necessarily simple) $\mathfrak{k}$ modules Z.

We now establish some general properties of the functors $R^*\Gamma$.

Proposition 2.6 *Suppose $M \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$ and W is a finite-dimensional $\mathfrak{g}$-module. Then for every $i \in \mathbb{N}$ we have a natural isomorphism $W \otimes_{\mathbb{C}} R^i\Gamma M \cong R^i\Gamma(W \otimes_{\mathbb{C}} M)$.*

Proof First we prove that if $N \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$, we have a natural isomorphism $W \otimes_{\mathbb{C}} \Gamma N \cong \Gamma(W \otimes_{\mathbb{C}} N)$: Since W is finite dimensional and ΓN is locally finite over $\mathfrak{k}$, we have a natural injective map from $W \otimes_{\mathbb{C}} \Gamma N$ into $\Gamma(W \otimes_{\mathbb{C}} N)$. Suppose Z is any finite-dimensional $\mathfrak{k}$-module. $\mathrm{Hom}_{\mathfrak{k}}(Z, -)$ is a left exact functor. Hence, we obtain a natural injective map

$$\alpha_Z : \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} \Gamma N) \to \mathrm{Hom}_{\mathfrak{k}}(Z, \Gamma(W \otimes_{\mathbb{C}} N)).$$

Now,

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} \Gamma N) &\cong \mathrm{Hom}_{\mathfrak{k}}(Z \otimes_{\mathbb{C}} W^*, \Gamma N) \\ &\cong \mathrm{Hom}_{\mathfrak{k}}(Z \otimes_{\mathbb{C}} W^*, N) \\ &\cong \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} N). \end{aligned}$$

Meanwhile, $\mathrm{Hom}_{\mathfrak{k}}(Z, \Gamma(W \otimes_{\mathbb{C}} N)) \cong \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} N)$. Hence, we have a commutative diagram, with vertical isomorphisms,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} \Gamma N) & \overset{\alpha_Z}{\hookrightarrow} & \mathrm{Hom}_{\mathfrak{k}}(Z, \Gamma(W \otimes_{\mathbb{C}} N)) \\ \downarrow_s & & \downarrow_s \\ \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} N) & \overset{\beta_Z}{\to} & \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} N), \end{array}$$

where β_Z is the map induced by α_Z.

We claim β_Z is the identity map. This follows from the canonical construction of the diagram. Hence, α_Z is an isomorphism. Since $W \otimes_{\mathbb{C}} \Gamma N$ and $\Gamma(W \otimes_{\mathbb{C}} N)$ are both locally finite over $\mathfrak{k}$, it follows that the injection of $W \otimes_{\mathbb{C}} \Gamma N$ into $\Gamma(W \otimes_{\mathbb{C}} N)$ is an isomorphism.

Next, we choose $i \in \mathbb{N}$ and $I^\bullet$ a resolution of M by injective objects in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$. We have $W \otimes_{\mathbb{C}} R^i\Gamma M \cong W \otimes_{\mathbb{C}} H^i(\Gamma I^\bullet) \cong H^i(W \otimes_{\mathbb{C}} \Gamma I^\bullet)$, since $W \otimes_{\mathbb{C}} (-)$ is an exact functor.

Thus, by the first part of the proof, $W \otimes_{\mathbb{C}} R^i\Gamma M \cong H^i(\Gamma(W \otimes_{\mathbb{C}} I^\bullet))$. Observe that $W \otimes_{\mathbb{C}} I^j$ is an injective object in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$: if Q is a module in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$, $\mathrm{Hom}_{\mathfrak{g}}(Q, W \otimes_{\mathbb{C}} I^j) \cong \mathrm{Hom}_{\mathfrak{g}}(Q \otimes_{\mathbb{C}} W^*, I^j)$; hence, $Q \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(Q, W \otimes_{\mathbb{C}} I^j)$ is an exact functor on $\mathcal{C}(\mathfrak{g},\mathfrak{m})$. Thus, $W \otimes_{\mathbb{C}} I^\bullet$ is a resolution of $W \otimes_{\mathbb{C}} M$ by injective objects in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$, and $H^i(\Gamma(W \otimes_{\mathbb{C}} I^\bullet)) \cong R^i\Gamma(W \otimes_{\mathbb{C}} M)$. $\Box$

For any $\mathfrak{g}$-module N, and any element $z \in Z_{U\mathfrak{g}}$, we write z_N for the $\mathfrak{g}$-module endomorphism of N defined by $z_N v = zv$ for $v \in N$.

Proposition 2.7 *Let $M \in \mathcal{C}(\mathfrak{g},\mathfrak{m})$. Then for every $i \in \mathbb{N}$, $z_{R^i\Gamma M} = R^i\Gamma z_M$.*

Proof Let $I^\bullet$ be a resolution of M in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$ by injective objects. The chain map $z_{I^\bullet}$ is a lifting of z_M to the resolution $I^\bullet$. The morphism $R^i\Gamma z_M$ is the action of z on $R^i\Gamma M$ induced by the chain map $z_{I^\bullet}$. Finally, the morphism $z_N : N \to N$ is natural in N. It follows that $R^i\Gamma z_M = z_{R^i\Gamma M}$. $\Box$

Now suppose $\mathfrak{a}$ is an ideal in $Z_{U\mathfrak{g}}$; for any $\mathfrak{g}$-module N, let $N^{\mathfrak{a}} = \{v \in N | \mathfrak{a}v = 0\}$.

Corollary 2.8 *If $M \in \mathcal{C}(\mathfrak{g},\mathfrak{m})$, and $M^{\mathfrak{a}} = M$, then $(R^*\Gamma M)^{\mathfrak{a}} = R^*\Gamma M$.*

Let us now return to the setup when $\mathfrak{k}$ is reductive in $\mathfrak{g}$.

Proof of Theorem 2.4 Let V be a finite-dimensional simple $\mathfrak{k}$-module. We will study $\mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma M)$. Let $I^\bullet$ be a resolution of M by injective objects in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$. By definition, $\mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma M) \cong \mathrm{Hom}_{\mathfrak{k}}(V, H^i(\Gamma I^\bullet))$; but $\mathrm{Hom}_{\mathfrak{k}}(V, -)$ is an exact functor in the category $\mathcal{C}(\mathfrak{g},\mathfrak{k})$, and hence
$\mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma M) \cong H^i(\mathrm{Hom}_{\mathfrak{k}}(V, \Gamma I^\bullet))$.

By Proposition 2.1, we have $\mathrm{Hom}_{\mathfrak{k}}(V, \Gamma J) \cong \mathrm{Hom}_{\mathfrak{k}}(V, J)$ for any $(\mathfrak{g},\mathfrak{m})$-module J. Hence, $\mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma M) \cong H^i(\mathrm{Hom}_{\mathfrak{k}}(V, I^\bullet))$.

Next, we observe that since I^j is an injective object in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$, then I^j is an injective object in $\mathcal{C}(\mathfrak{k},\mathfrak{m})$. To see this, let N be an object in $\mathcal{C}(\mathfrak{k},\mathfrak{m})$. $\mathrm{Hom}_{\mathfrak{k}}(N, I^j) \cong \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes_{U\mathfrak{k}} N, I^j)$. Note that $U\mathfrak{g} \otimes_{U\mathfrak{k}} N$ is an object in $\mathcal{C}(\mathfrak{g},\mathfrak{m})$. The functor $U\mathfrak{g} \otimes_{U\mathfrak{k}} (-)$ is exact. Also, the functor $\mathrm{Hom}_{\mathfrak{g}}(-, I^j)$ from $\mathcal{C}(\mathfrak{g},\mathfrak{m})$ to $\mathbb{C}$-mod is exact. Hence, by the above natural isomorphism, the functor $\mathrm{Hom}_{\mathfrak{k}}(-, I^j)$ is exact. Therefore I^j is an injective object in $\mathcal{C}(\mathfrak{k},\mathfrak{m})$, and $I^\cdot$ is an injective resolution of M in $\mathcal{C}(\mathfrak{k},\mathfrak{m})$.

We now conclude that $\mathrm{Hom}_{\mathfrak{k}}(V, R^*\Gamma M) \cong \mathrm{Ext}^*_{\mathcal{C}(\mathfrak{k},\mathfrak{m})}(V, M)$.

If $\mathfrak{m} = 0$, classical homological algebra tells us that $\mathrm{Ext}^*_{\mathfrak{k}}(V, M) \cong H^*(\mathfrak{k}, V^* \otimes_{\mathbb{C}} M)$, where $H^*(\mathfrak{k}, -)$ is Lie algebra cohomology [BW]. Since $\mathfrak{m}$ is reductive in $\mathfrak{k}$, we have

$$\mathrm{Ext}^*_{C(\mathfrak{k},\mathfrak{m})}(V, M) \cong H^*(\mathfrak{k}, \mathfrak{m}, V^* \otimes_{\mathbb{C}} M) = H^*(\mathrm{Hom}_{\mathfrak{m}}(\Lambda^\bullet(\mathfrak{k}/\mathfrak{m}), V^* \otimes_{\mathbb{C}} M)),$$

where the complex $\mathrm{Hom}_{\mathfrak{m}}(\Lambda^\bullet(\mathfrak{k}/\mathfrak{m}), V^* \otimes_{\mathbb{C}} M)$ is endowed with the relative Koszul differential.

Part b) of Theorem 2.4 is now immediate, since $\Lambda^i(\mathfrak{k}/\mathfrak{m}) = 0$ for $i > \dim(\mathfrak{k}/\mathfrak{m})$; thus, for any simple finite-dimensional $\mathfrak{k}$-module V,

$$\mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma M) = 0 \text{ for } i > \dim(\mathfrak{k}/\mathfrak{m}).$$

By assumption, M has finite type over $\mathfrak{m}$, V is finite dimensional and $\Lambda^i(\mathfrak{k}/\mathfrak{m})$ is finite dimensional. Hence,

$$\dim \mathrm{Hom}_{\mathfrak{m}}(\Lambda^i(\mathfrak{k}/\mathfrak{m}), V^* \otimes M) = \dim \mathrm{Hom}_{\mathfrak{m}}(V \otimes_{\mathbb{C}} \Lambda^i(\mathfrak{k}/\mathfrak{m}), M) < \infty.$$

From the isomorphisms proved above we conclude that part a) holds. Finally, parts a) and b) imply part c). □

Definition 2.9 *A $\mathfrak{g}$-module N is locally $Z_{U\mathfrak{g}}$-finite if for any $v \in N$, $Z_{U\mathfrak{g}}v$ is finite dimensional.*

If $\theta : Z_{U\mathfrak{g}} \to \mathbb{C}$ is a homomorphism and N is a $\mathfrak{g}$-module, we set $P_\theta N = \cup_{s\in\mathbb{N}} N^{(\mathrm{Ker}\theta)^s}$. Observe that $P_\theta N$ is a $\mathfrak{g}$-submodule of N. By C we denote the set of homomorphisms of $Z_{U\mathfrak{g}}$ to $\mathbb{C}$ (central characters).

Lemma 2.10 *If a $U\mathfrak{g}$-module N is locally $Z_{U\mathfrak{g}}$-finite, then $N = \bigoplus_{\theta\in C} P_\theta N$.*

Proof By definition, $\bigoplus_{\theta\in C} P_\theta N \subset N$. To show the lemma, note that for any $v \in N$, $Z_{U\mathfrak{g}}v$ is a finite-dimensional $Z_{U\mathfrak{g}}$-submodule of N. By decomposing v as a sum of generalized $Z_{U\mathfrak{g}}$-eigenvectors, we obtain $v \in \bigoplus_{\theta\in C} P_\theta N$. □

Proposition 2.11 *$R^i\Gamma$ commutes with inductive limits.*

Proof Let V be a finite-dimensional simple $\mathfrak{k}$-module. By the proof of Theorem 2.4, we have a natural isomorphism
$\mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma M) \cong H^i(\mathrm{Hom}_{\mathfrak{m}}(\Lambda^\bullet(\mathfrak{k}/\mathfrak{m}), V^* \otimes_{\mathbb{C}} M))$. Since $\Lambda^\bullet(\mathfrak{k}/\mathfrak{m})$ is finite dimensional, the functor $H^i(\mathrm{Hom}_{\mathfrak{m}}(\Lambda^\bullet(\mathfrak{k}/\mathfrak{m}), V^* \otimes_{\mathbb{C}} (-)))$ commutes with inductive limits. Hence, the functor $\mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma)$ commutes with inductive limits. Finally, $R^i\Gamma \cong \bigoplus_V \mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma)$, where V runs over finite-dimensional simple $\mathfrak{k}$-modules. □

Proposition 2.12 *If $M \in C(\mathfrak{g}, \mathfrak{m})$ is locally $Z_{U\mathfrak{g}}$-finite, then so is $R^i\Gamma M$ for any i. Moreover, for any $\theta \in C$, $P_\theta(R^i\Gamma M) = R^i\Gamma(P_\theta M)$.*

Proof If $M \in C(\mathfrak{g}, \mathfrak{m})$ is locally $Z_{U\mathfrak{g}}$-finite, then M is an inductive limit of submodules annihilated by an ideal of finite codimension in $Z_{U\mathfrak{g}}$. By Corollary 2.8 and Proposition 2.11, $R^i\Gamma M$ is likewise an inductive limit of modules annihilated by an ideal of finite codimension in $Z_{U\mathfrak{g}}$. Hence, $R^i\Gamma M$ is locally $Z_{U\mathfrak{g}}$-finite. Moreover, Corollary 2.8 and Lemma 2.10 allow us to conlude that $P_\theta(R^i\Gamma M) = R^i\Gamma(P_\theta M)$ for any $\theta \in C$. □

Proposition 2.13 *For $M \in \mathcal{A}(\mathfrak{g}, \mathfrak{m})$ and $V \in \mathrm{Rep}\,\mathfrak{k}$ we have*

$$\sum_i (-1)^i \dim \mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma M) = \sum_i (-1)^i \dim \mathrm{Hom}_{\mathfrak{m}}(V \otimes_{\mathbb{C}} \Lambda^i(\mathfrak{k}/\mathfrak{m}), M).$$

In particular, the alternating sum on the left depends only on the restriction of M to $\mathfrak{m}$.

Proof By the proof of Theorem 2.4 we have

$$\dim \mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma M) = \dim H^i(\mathrm{Hom}_{\mathfrak{m}}(V \otimes_{\mathbb{C}} \Lambda^\bullet(\mathfrak{k}/\mathfrak{m}), M)) < \infty.$$

The proof of Proposition 2.13 now follows from the Euler-Poincare principle. □

Remark. The dimension of $\mathrm{Hom}_{\mathfrak{k}}(V, R^i\Gamma M) \cong H^i(\mathfrak{k}, \mathfrak{m}, V^* \otimes_{\mathbb{C}} M)$ will in general depend on the $\mathfrak{g}$-module M and not just its restriction to $\mathfrak{m}$.

Remark. Theorem 2.4 b) holds for $M \in C(\mathfrak{g}, \mathfrak{m})$. In particular, we obtain the following.

Corollary 2.14 *If* $M \in C(\mathfrak{g}, \mathfrak{m})$ *and* $i > \dim \mathfrak{k}/\mathfrak{m}$, *then*

$$H^i(\Gamma\mathrm{Hom}(K_\bullet(\mathfrak{g}, \mathfrak{m}), M)) = 0.$$

Note that $K_i(\mathfrak{g}, \mathfrak{m}) = 0$ if and only if $i > \dim(\mathfrak{g}/\mathfrak{m})$.

Even if M is simple over $U\mathfrak{g}$, we can have $R^p\Gamma M$ reducible over $U\mathfrak{g}$. We don't have to look hard for examples.

Example 2.15

Recall that $\mathfrak{m}$ is reductive in $\mathfrak{k}$. Let $n = \dim(\mathfrak{k}/\mathfrak{m})$. Then $H^n(\mathfrak{k}, \mathfrak{m}, \mathbb{C}) \cong \mathbb{C}$. Hence, $R^n\Gamma\mathbb{C} \cong \mathbb{C}$. Thus, the inequality for the vanishing of $R^i\Gamma M$ in Theorem 2.4 b) is sharp.

Example 2.16

Assume $\mathfrak{m} = \mathfrak{t}$, a Cartan subalgebra of $\mathfrak{k}$. Let K be a connected complex algebraic group with Lie algebra $\mathfrak{k}$, and let T be the subgroup of K with Lie algebra $\mathfrak{t}$. Let K_0 be a maximal compact subgroup of K; choose K_0 so that $K_0 \cap T = T_0$ is a maximal torus in T. We have $H^*(K_0/T_0) \cong H^*(\mathfrak{k}, \mathfrak{t}, \mathbb{C})$. If $l = \mathrm{rk}\mathfrak{k}_{ss}$, then $\dim H^2(K_0/T_0) = l$. Now, as a $\mathfrak{k}$-module, $R^*\Gamma\mathbb{C} \cong H^*(\mathfrak{k}, \mathfrak{m}, \mathbb{C})$ with the trivial action of $\mathfrak{k}$. Hence $\dim R^2\Gamma\mathbb{C} = l$, and thus $R^2\Gamma\mathbb{C}$ is in general a reducible trivial $\mathfrak{k}$-module. In fact, $R^2\Gamma\mathbb{C}$ is in general a reducible trivial $\mathfrak{g}$-module.

Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and extend it to be a Cartan subalgebra $\mathfrak{h}$ on $\mathfrak{g}$. Let M be a simple module in $\mathcal{A}(\mathfrak{g}, \mathfrak{t})$. Then M is automatically in $\mathcal{A}(\mathfrak{g}, \mathfrak{h})$. Simple modules in $\mathcal{A}(\mathfrak{g}, \mathfrak{h})$ have been classified by S. Fernando and O. Mathieu, see [F] and [M]. It is an open problem to determine which simple $(\mathfrak{g}, \mathfrak{h})$-modules of finite type over $\mathfrak{h}$ are also of finite type over $\mathfrak{t}$. We want to study $R^*\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{t}}M$ when M is a module in $\mathcal{A}(\mathfrak{g}, \mathfrak{t})$.

Consider first the case when $\mathfrak{h} = \mathfrak{t}$, i.e. where $\mathfrak{k}$ is a root subalgebra of $\mathfrak{g}$. As an interesting exercise, for the example $\mathfrak{g} = \mathfrak{sl}(n)$ and M a Britten-Lemire module, one can show that $R^*\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{h}}M = 0$. If we take some Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $\mathfrak{h}$, and choose a weight $\lambda \in \mathfrak{h}^*$, we can study $R^*\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{h}}\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}\mathbb{C}_\lambda$. This is a family of graded $(\mathfrak{g}, \mathfrak{k})$-modules in $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$. Let us examine the behavior of these $(\mathfrak{g}, \mathfrak{k})$-modules in some examples.

Example 2.17

Suppose $\mathfrak{k} = \mathfrak{g}$, then $R^*\Gamma\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}\mathbb{C}_\lambda \in \mathcal{A}(\mathfrak{g},\mathfrak{g})^*$. Either all the derived functor modules vanish, or for exactly one degree, say $i(\lambda)$, $R^{i(\lambda)}\Gamma\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}\mathbb{C}_\lambda \neq 0$ and is a simple $\mathfrak{g}$-module. This is the situation considered in the Borel-Weil-Bott theorem, see [EW].

Example 2.18

Let $\mathfrak{g} = \mathfrak{sl}(n)$, $n = p+q$ with $p, q > 0$, $\mathfrak{k} = \mathfrak{s}(\mathfrak{gl}(p) \oplus \mathfrak{gl}(q)) := \{m \oplus n \mid \mathrm{tr}m = -\mathrm{tr}n\}$, and $\mathfrak{h}$ the diagonal Cartan subalgebra. Choose some Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$, and also choose $\lambda \in \mathfrak{h}^*$. Let $s = \frac{1}{2}\dim \mathfrak{k}/\mathfrak{h}$. Then,

$$A_{\mathfrak{b}}(\lambda) := R^s\Gamma\ \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}\ \mathbb{C}_\lambda$$

can either be 0 or an infinite-dimensional $(\mathfrak{g},\mathfrak{k})$-module [VZ]. It may be simple or reducible, and if reducible, it may not be semisimple over $U\mathfrak{g}$. What are the possibilities for $\mathfrak{g}[A_{\mathfrak{b}}(\lambda)]$? There are three possibilities:

1. $\mathfrak{g}[A_{\mathfrak{b}}(\lambda)] = \mathfrak{k}$
2. $\mathfrak{g}[A_{\mathfrak{b}}(\lambda)] = \mathfrak{k} \niplus \tilde{\mathfrak{n}}^+$
3. $\mathfrak{g}[A_{\mathfrak{b}}(\lambda)] = \mathfrak{k} \niplus \tilde{\mathfrak{n}}^-$

where $\tilde{\mathfrak{n}}^+$ is the nilradical for a maximal parabolic containing $\mathfrak{k}$, and $\tilde{\mathfrak{n}}^-$ is its opposite. Conversely, for each of these choices, we can give a pair $(\mathfrak{b},\lambda)$ so that $\mathfrak{g}[A_{\mathfrak{b}}(\lambda)]$ is that subalgebra. For this fact, see [PZ4].

Consider now the general case when $\mathfrak{h} \neq \mathfrak{t}$. Let $\mathfrak{g}$ be any semisimple Lie algebra and, in addition to $\mathfrak{t}$ and $\mathfrak{k}$, consider an arbitrary parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ with $\mathfrak{p} \supset \mathfrak{h}$. Let $N \in \mathcal{A}(\mathfrak{p},\mathfrak{t})$ and consider $R^*\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{t}}\ \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}N$. Note that $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}N$ may or may not be in $\mathcal{A}(\mathfrak{g},\mathfrak{t})$. The parabolic subalgebra $\mathfrak{p}$ has a Levi decomposition $\mathfrak{p} = \mathfrak{p}_{red} \niplus \mathfrak{n}_{\mathfrak{p}}$, and as a vector space we can write $\mathfrak{g} = \mathfrak{n}_{\mathfrak{p}}^- \oplus \mathfrak{p}_{red} \oplus \mathfrak{n}_{\mathfrak{p}}$. Then, $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}N \simeq U(\mathfrak{n}_{\mathfrak{p}}^-) \otimes_{\mathbb{C}} N$. In this presentation, $\mathfrak{t}$ can be seen to act by the adjoint action on $\mathfrak{n}_{\mathfrak{p}}^-$. There are choices where $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}N$ is not in $\mathcal{A}(\mathfrak{g},\mathfrak{t})$, but $R^*\Gamma\ \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}N \in \mathcal{A}(\mathfrak{g},\mathfrak{k})$.

For example, let $\mathfrak{l}$ be simple, $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}$, $\mathfrak{k}$ the diagonal embedding of $\mathfrak{l}$ into $\mathfrak{g}$. Let $\mathfrak{t}$ be the Cartan of $\mathfrak{k}$ and $\mathfrak{p}$ the sum of a Borel of $\mathfrak{l}$ in the first factor with the opposite Borel in the second factor.

Definition 2.19 *Given a triple* $(\mathfrak{g},\mathfrak{k},\mathfrak{t})$*, a* $\mathfrak{t}$*-compatible parabolic subalgebra is any parabolic subalgebra of the form*

$$\mathfrak{p}_h = \bigoplus_{\mathrm{Re}\ \alpha(h) \geq 0} \mathfrak{g}_\alpha, \tag{2.5}$$

for a fixed $h \in \mathfrak{t}$*.*

In this definition, $\mathfrak{g}_\alpha$ is the α-weight space for $\mathfrak{t}$ acting on $\mathfrak{g}$. Also, $\mathfrak{t} \subseteq \mathfrak{g}_0 \subseteq \mathfrak{p}_h$, $\mathfrak{p}_{h,red} = \bigoplus_{\mathrm{Re}\ \alpha(h)=0} \mathfrak{g}_\alpha$, and $\mathfrak{n}_{\mathfrak{p}_h} = \bigoplus_{\mathrm{Re}\ \alpha(h)>0} \mathfrak{g}_\alpha$

Lemma 2.20 *Let $\mathfrak{p}$ be a $\mathfrak{t}$-compatible parabolic subalgebra. Assume $\mathfrak{p} = \mathfrak{p}_h$ and h acts by a scalar in $N \in \mathcal{A}(\mathfrak{p}, \mathfrak{t})$; then* $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N \in \mathcal{A}(\mathfrak{g}, \mathfrak{t})$.

Proof By the same argument as in the proof of Lemma 1.5, if N is a $(\mathfrak{p}, \mathfrak{t})$-module, then $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$ is a $(\mathfrak{g}, \mathfrak{t})$-module. Moreover, by the same argument as in the proof of Lemma 1.7, $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N \cong S(\mathfrak{g}/\mathfrak{p}) \otimes_{\mathbb{C}} N$ as a $(\mathfrak{t}, \mathfrak{t})$-module. By the assumption that $\mathfrak{p} = \mathfrak{p}_h$, the eigenspaces of h in $S(\mathfrak{g}/\mathfrak{p})$ are finite dimensional. By the assumption on N, h acts by a scalar in N. It follows that the weight spaces of $\mathfrak{t}$ in $S(\mathfrak{g}/\mathfrak{p}) \otimes_{\mathbb{C}} N$ are finite dimensional. □

Consider $R^*\Gamma \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$ for $\mathfrak{p}$ as above. This is a generalized Harish-Chandra module, but we also have a vanishing theorem which tells us $R^i\Gamma \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N = 0$ if $i < s := \frac{1}{2} \dim \mathfrak{k}/\mathfrak{t}$. This, along with an earlier result, indicates that the only possibly nonvanishing derived functors occur for $s \leq i \leq 2s$. The proof of vanishing is given in [V, Ch.6]; see also [PZ3]. This construction is known as cohomological induction, see also [KV].

Consider now the special case when $\mathfrak{k}$ is isomorphic to $\mathfrak{sl}(2)$. There are finitely many conjugacy classes of such subalgebras, classified by Dynkin [Dy]. Examples include:

- $\mathfrak{g} = \mathfrak{sl}(3)$, $\mathfrak{k} = \mathfrak{so}(3) \simeq \mathfrak{sl}(2)$; this pair is symmetric.
- $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, $\mathfrak{k} = \mathfrak{sl}(2) = \{(Y, Y) \in \mathfrak{g} | Y \in \mathfrak{sl}(2)\}$; this pair is symmetric.
- $\mathfrak{g} = \mathfrak{sp}(4)$, $\mathfrak{k} = \mathfrak{sl}(2)$, the principal $\mathfrak{sl}(2)$;this pair is not symmetric.
- $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, with $\mathfrak{k} = \mathfrak{sl}(2) = \{(Y, Y, Y) \in \mathfrak{g} |\ Y \in \mathfrak{sl}(2)\}$; this pair is not symmetric.

Since $\mathfrak{t}$ is one-dimensional when $\mathfrak{k} = \mathfrak{sl}(2)$, we have $s = 1$, so we need only study $i = 1, 2$. The duality theorem for relative Lie algebra cohomology [BW] implies that if $\Gamma\ (\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N)^* = 0$, then $R^2\Gamma \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N = 0$. So, we need only study $R^1\Gamma\ \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$. This module may still be 0. We will try to understand $R^1\Gamma \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$ by using an Euler characteristic trick. Given a module V for $\mathfrak{k} \simeq \mathfrak{sl}(2)$, we have

$$\dim\ \mathrm{Hom}_{\mathfrak{k}}(V, R^1\Gamma\ \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N) = -\sum_j (-1)^j \dim\ \mathrm{H}^j(\mathfrak{k}, \mathfrak{t}; V^* \otimes_{\mathbb{C}} \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N). \qquad (2.6)$$

This formula is an immediate consequence of Proposition 2.13. We should think of (2.6) as analogous to the study of cohomology of surfaces with coefficients in locally constant sheaves. Using formula (2.6) one can in specific cases show $R^1\Gamma\ \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$ is nonzero.

Since $\dim\ \mathfrak{t} = 1$, $\mathfrak{p} = \mathfrak{p}_h$ is one of two parabolic subalgebras. In either case, $(\mathfrak{p}_h)_{red} \simeq C_{\mathfrak{g}}(\mathfrak{t})$ and the nilpotent part depends on the decomposition of $\mathfrak{g}$ into weights under the action of $\mathfrak{t}$. In general, $\mathfrak{p}$ is not a Borel subalgebra. Let N be a finite dimensional module over $C_{\mathfrak{g}}(\mathfrak{t})$, with trivial action of $\mathfrak{n}_{\mathfrak{p}}$. Consider $R^1\Gamma\ \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$. We can use formula (12) to compute the $\mathfrak{k}$-multiplicities as a sequence of natural numbers, see [PZ1].

3 Construction and reconstruction of generalized Harish-Chandra modules

In this final section we give an introduction to the results of [PZ3]. These results provide a classification of a class of simple generalized Harish-Chandra modules.

We start by introducing a notation. A *multiset* is a function f from a set D into $\mathbb{N}$. A *submultiset* of f is a multiset f' defined on the same domain D such that $f'(d) \leq f(d)$ for any $d \in D$. For any finite multiset f, defined on an additive monoid D, we can put $\rho_f := \frac{1}{2} \sum\limits_{d \in D} f(d)d$.

We assume that the quadruple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \mathfrak{t})$ as in the previous section is fixed. We assume further that $\mathfrak{k}$ is algebraic in $\mathfrak{g}$. If $M = \bigoplus_{\omega \in \mathfrak{t}^*} M(\omega)$ is a $\mathfrak{t}$-weight module for which all $M(\omega)$ are finite dimensional, M determines the multiset $\mathrm{ch}_{\mathfrak{t}} M$ which is the function $\omega \mapsto \dim M(\omega)$ defined on the set of $\mathfrak{t}$-weights of M.

Note that the $\mathbb{R}$-span of the roots of $\mathfrak{h}$ in $\mathfrak{g}$ fixes a real structure on $\mathfrak{h}^*$, whose projection onto $\mathfrak{t}^*$ is a well-defined real structure on $\mathfrak{t}^*$. In what follows, we will denote by $\mathrm{Re}\lambda$ the real part of an element $\lambda \in \mathfrak{t}^*$. We fix also a Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{k}$ with $\mathfrak{b}_{\mathfrak{k}} \supset \mathfrak{t}$. Then $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{t} \niplus \mathfrak{n}_{\mathfrak{k}}$, where $\mathfrak{n}_{\mathfrak{k}}$ is the nilradical of $\mathfrak{b}_{\mathfrak{k}}$. We set $\rho := \rho_{\mathrm{ch}_{\mathfrak{t}} \mathfrak{n}_{\mathfrak{k}}}$, and $\rho_{\mathfrak{n}}^{\perp} = \rho_{\mathrm{ch}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}^{\perp})}$.

Let $\langle \, , \rangle$ be the unique $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}^*$ such that $\langle \alpha, \alpha \rangle = 2$ for any long root of a simple component of $\mathfrak{g}$. The form $\langle \, , \rangle$ enables us to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then $\mathfrak{h}$ is identified with $\mathfrak{h}^*$, and $\mathfrak{k}$ is identified with $\mathfrak{k}^*$. The superscript $\perp$ indicates orthogonal space. Note that there is a canonical $\mathfrak{k}$-module decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^{\perp}$. We also set $\|\kappa\|^2 := \langle \kappa, \kappa \rangle$ for any $\kappa \in \mathfrak{h}^*$.

We say that an element $\lambda \in \mathfrak{t}^*$ is $(\mathfrak{g}, \mathfrak{k})$-*regular* if $\langle \mathrm{Re}\lambda, \alpha \rangle \neq 0$ for non-zero $\mathfrak{t}$-weights α of $\mathfrak{g}$. Since we identify $\mathfrak{t}$ with $\mathfrak{t}^*$, we can consider $\mathfrak{t}$-compatible parabolic subalgebras $\mathfrak{p}_{\lambda}$ for $\lambda \in \mathfrak{t}^*$.

By $\mathfrak{m}_{\lambda}$ and $\mathfrak{n}_{\lambda}$ we denote respectively the reductive part of $\mathfrak{p}_{\lambda}$ (containing $\mathfrak{h}$) and the nilradical of $\mathfrak{p}_{\lambda}$. A $\mathfrak{t}$-compatible parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \niplus \mathfrak{n}$ (i.e. $\mathfrak{p} = \mathfrak{p}_{\lambda}$ for some $\lambda \in \mathfrak{t}^*$) is *minimal* if it does not properly contain another $\mathfrak{t}$-compatible parabolic subalgebra. It is easy to see that a $\mathfrak{t}$-compatible parabolic subalgebra $\mathfrak{p}_{\lambda}$ is minimal if and only if $\mathfrak{m}_{\lambda}$ equals the centralizer $C_{\mathfrak{g}}(\mathfrak{t})$, or equivalently if and only if λ is $(\mathfrak{g}, \mathfrak{k})$-regular.

A $\mathfrak{k}$-*type* is by definition a simple finite-dimensional $\mathfrak{k}$-module. By V_{μ} we will denote a $\mathfrak{k}$-type with $\mathfrak{b}_{\mathfrak{k}}$-highest weight μ (μ is then $\mathfrak{k}$-integral and $\mathfrak{b}_{\mathfrak{k}}$-dominant). If M is a $(\mathfrak{g}, \mathfrak{k})$-module and V_{μ} is a $\mathfrak{k}$-type, let $M[\mu]$ denote the V_{μ}-isotypic $\mathfrak{k}$-submodule of M. (See the discussion after equation (1.6) in Section 1.) Let V_{μ} be a $\mathfrak{k}$-type such that $\mu + 2\rho$ is $(\mathfrak{g}, \mathfrak{k})$-regular, and let $\mathfrak{p} = \mathfrak{m} \niplus \mathfrak{n}$ be the $\mathfrak{t}$-compatible parabolic subalgebra $\mathfrak{p}_{\mu + 2\rho}$. Note that $\mathfrak{p}$ is a minimal $\mathfrak{t}$-compatible parabolic subalgebra. Put $\rho_{\mathfrak{n}} := \rho_{\mathrm{ch}_{\mathfrak{t}} \mathfrak{n}}$.

The following is a key definition. We say that V_{μ} is *generic* if the following two conditions hold:

(1) $\langle \mathrm{Re}\mu + 2\rho - \rho_{\mathfrak{n}}, \alpha \rangle \geq 0$ for every $\mathfrak{t}$-weight α of $\mathfrak{n}_{\mathfrak{k}}$.

(2) $\langle \mathrm{Re}\mu + 2\rho - \rho_S, \rho_S \rangle > 0$ for every submultiset S of $\mathrm{ch}_{\mathfrak{t}} \mathfrak{n}$.

One can show that the following is a sufficient condition for genericity: $|\langle \mathrm{Re}\mu + 2\rho, \alpha\rangle| \geq c$ for any $\mathfrak{t}$-weight α of $\mathfrak{g}$ and a suitably large positive constant c, depending only on the pair $(\mathfrak{g}, \mathfrak{k})$.

Let $\Theta_{\mathfrak{k}}$ be the discrete subgroup of $Z(\mathfrak{k})^*$ generated by $\mathrm{supp}_{Z(\mathfrak{k})}\mathfrak{g}$. By $\mathcal{M}$ we denote the class of $(\mathfrak{g}, \mathfrak{k})$-modules M for which there exists a finite subset $S \subset Z(\mathfrak{k})^*$ such that $\mathrm{supp}_{Z(\mathfrak{k})}M \subset (S + \Theta_{\mathfrak{k}})$. Note that any finite length $(\mathfrak{g}, \mathfrak{k})$-module lies in the class $\mathcal{M}$.

If M is a module in $\mathcal{M}$, a $\mathfrak{k}$-type V_μ of M is *minimal* if the function $\mu' \mapsto \|\mathrm{Re}\mu' + 2\rho\|^2$ defined on the set $\{\mu' \in \mathfrak{t}^* \mid M[\mu'] \neq 0\}$ has a minimum at μ. Any non-zero $(\mathfrak{g}, \mathfrak{k})$-module M in $\mathcal{M}$ has a minimal $\mathfrak{k}$-type. This follows from the fact that the squared length of a vector has a minimum on every shifted lattice in Euclidean space.

We need also the following "production" or "coinduction" functor (see [Bla]) from the category of $(\mathfrak{p}, \mathfrak{t})$-modules to the category of $(\mathfrak{g}, \mathfrak{t})$-modules:

$$\mathrm{pro}_{\mathfrak{p},\mathfrak{t}}^{\mathfrak{g},\mathfrak{t}}(N) := \Gamma_{\mathfrak{t},0}(\mathrm{Hom}_{U\mathfrak{p}}(U\mathfrak{g}, N)).$$

The functor $\mathrm{pro}_{\mathfrak{p},\mathfrak{t}}^{\mathfrak{g},\mathfrak{t}}$ is exact.

Definition 3.1 *Let $\mathfrak{p} = \mathfrak{m} \ni \mathfrak{n}$ be a minimal $\mathfrak{t}$-compatible parabolic subalgebra, E be a simple finite-dimensional $\mathfrak{p}$-module on which $\mathfrak{t}$ acts via a weight ω. We call the series of $(\mathfrak{g}, \mathfrak{k})$-modules of finite type*

$$F^*(\mathfrak{p}, E) := R^*\Gamma_{\mathfrak{k},\mathfrak{t}}(\mathrm{pro}_{\mathfrak{p},\mathfrak{t}}^{\mathfrak{g},\mathfrak{t}}(E \otimes_{\mathbb{C}} \Lambda^{\dim \mathfrak{n}}(\mathfrak{n}))),$$

the fundamental series of generalized Harish-Chandra modules.

Set $\mu := \omega + 2\rho_{\mathfrak{n}}^{\perp}$. It is proved in [PZ2] that the following assertions hold under the assumptions that $\mathfrak{p} \subseteq \mathfrak{p}_{\mu+2\rho}$ and that μ is $\mathfrak{b}_{\mathfrak{k}}$-dominant and $\mathfrak{k}$-integral.

a) $F^*(\mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k})$-module of finite type in the class $\mathcal{M}$.

b) There is a $\mathfrak{k}$-module isomorphism

$$F^s(\mathfrak{p}, E)[\mu] \cong \mathbf{C}^{\dim E} \otimes_{\mathbb{C}} V_\mu,$$

and V_μ is the unique minimal $\mathfrak{k}$-type of $F^s(\mathfrak{p}, E)$.

c) Let $\bar{F}^s(\mathfrak{p}, E)$ be the $\mathfrak{g}$-submodule of $F^s(\mathfrak{p}, E)$ generated by $F^s(\mathfrak{p}, E)[\mu]$. Then any simple quotient of $\bar{F}^s(\mathfrak{p}, E)$ has minimal $\mathfrak{k}$-type V_μ.

The following theorems provide the basis of the classification of $(\mathfrak{g}, \mathfrak{k})-$ modules with generic minimal $\mathfrak{k}$-type. The classification is then stated as a corollary.

Theorem 3.2 *(First reconstruction theorem, [PZ3]) Let M be a simple $(\mathfrak{g}, \mathfrak{k})$-module of finite type with a minimal $\mathfrak{k}$-type V_μ which is generic. Then $\mathfrak{p} := \mathfrak{p}_{\mu+2\rho} = \mathfrak{m} \ni \mathfrak{n}$ is a minimal $\mathfrak{t}$-compatible parabolic subalgebra. Let E be the $\mathfrak{p}$-module $H^r(\mathfrak{n}, M)(\mu - 2\rho_{\mathfrak{n}}^{\perp})$ with trivial $\mathfrak{n}$-action, where $r = \dim(\mathfrak{n} \cap \mathfrak{k}^{\perp})$. Then E is a simple $\mathfrak{p}$-module and M is canonically isomorphic to $\bar{F}^s(\mathfrak{p}, E)$ for $s = \dim(\mathfrak{n} \cap \mathfrak{k})$.*

Theorem 3.3 *(Second reconstruction theorem, [PZ3]) Assume that the pair* $(\mathfrak{g}, \mathfrak{k})$ *is regular, i.e.* $\mathfrak{t}$ *contains a regular element of* $\mathfrak{g}$*. Let* M *be a simple* $(\mathfrak{g}, \mathfrak{k})$*-module (a priori of infinite type) with a minimal* $\mathfrak{k}$*-type* V_μ *which is generic. Then* M *has finite type, and hence by Theorem 3.2,* M *is canonically isomorphic to* $\bar{F}^s(\mathfrak{p}, E)$ *(where* $\mathfrak{p}, E$ *and* s *are as in Theorem 3.2).*

Corollary 3.4 *(Classification of generalized Harish-Chandra modules with generic minimal* $\mathfrak{k}$*-type.) Fix a generic* $\mathfrak{k}$*-type* V_μ*. The simple* $(\mathfrak{g}, \mathfrak{k})$*-modules of finite type whose minimal* $\mathfrak{k}$*-type is isomorphic to* V_μ *are in a natural bijective correspondence with the simple finite-dimensional* $\mathfrak{p}_{\mu+2\rho}$*-modules on which* $\mathfrak{t}$ *acts via* $\mu - 2\rho_{\mathfrak{n}}^{\perp}$.

Note that if $\mathrm{rk}\mathfrak{k} = \mathrm{rk}\mathfrak{g}$, there exists a unique simple finite-dimensional $\mathfrak{p}_{\mu+2\rho}$-module E on which $\mathfrak{t}$ acts via $\mu - 2\rho_{\mathfrak{n}}^{\perp}$. If $\mathrm{rk}\mathfrak{g} - \mathrm{rk}\mathfrak{k} = d > 0$, there exists a d-parameter family of such $\mathfrak{p}_{\mu+2\rho}$-modules.

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