Parameters for twisted representations

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The first author dedicates this to the second on the occasion of his 60th birthday

Abstract The main result of [4] is the description of an algorithm to compute the signature of the Hermitian form on an irreducible representation of a real reductive Lie group G, and therefore determine if it is unitary. This paper concerns an important ingredient of the algorithm. If the inner class of G is defined by an outer automorphism δ, so that G does not have discrete series representations, it is necessary to compute a new class of Kazhdan–Lusztig–Vogan polynomials for G. These were defined and studied by Lusztig and Vogan in [10]. In order to carry out the computation, we introduce new class of *twisted* parameters, and study the Hecke algebra action in the resulting basis.

Key words: unitary representation, Kazhdan–Lusztig polynomial, Hermitian form

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1 Introduction

One of the central problems in representation theory is understanding irreducible unitary representations. The reason is that in many applications of linear algebra (like those of representation theory to harmonic analysis) the notion of *length* of vectors is fundamentally important. Unitary representations are exactly those preserving a good notion of length.

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The paper [4] provides an algorithm for calculating the irreducible unitary representations of a real reductive Lie group G. The starting point for this algorithm is the Langlands classification, which provides a parameter space for the irreducible admissible representations of G. In order to determine the unitary representations, it is necessary to pass to a larger extended group ${}^{\delta}G$ containing G of index 2, and construct a parameter space for the representations of ${}^{\delta}G$. The purpose of this paper is to address the following problem: when a parameter for G extends to ${}^{\delta}G$ in two ways, there is no canonical way to choose one of the extensions. Consequently the theory for G does not carry over to ${}^{\delta}G$ in a simple way, and it is necessary to define parameters for ${}^{\delta}G$ and study their properties in some detail.

In order to explain this we need to describe briefly (or at least more briefly than [4]) the nature of the unitarity algorithm. In order to minimize technicalities, we will provide in the introduction complete details only for *finite-dimensional* representations. For a real reductive Lie group, the theory of Harish-Chandra modules provides a complete way to deal with the complications attached to infinite-dimensional representations.

To study unitary representations it is natural to study the larger class of representations with invariant Hermitian forms. Here is the underlying formalism.

Definition 1.1. Suppose V and W are complex vector spaces. A *sesquilinear pairing* is a map

$$\langle\cdot,\cdot\rangle\colon V\times W\to\mathbb{C}$$

that is linear in V and conjugate-linear in W:

$$\langle av_1+bv_2,w\rangle=a\langle v_1,w\rangle+b\langle v_2,w\rangle,\quad \langle v,cw_1+dw_2\rangle=\overline{c}\langle v,w_1\rangle+\overline{d}\langle v,w_2\rangle.$$

In case $V=W$, the pairing is called *Hermitian* if in addition

$$\langle v_1,v_2\rangle=\overline{\langle v_2,v_1\rangle}.$$

If $\langle,\rangle$ is a *nondegenerate* Hermitian pairing on a *finite-dimensional* vector space V, then there is a one-to-one correspondence between linear maps $A\in\mathrm{Hom}(V,V)$ and sesquilinear pairings $\langle,\rangle_A$ on V, defined by

$$\langle v,w\rangle_A=\langle v,Aw\rangle.$$

In this correspondence, $\langle,\rangle_A$ is Hermitian if and only if A is self-adjoint with respect to $\langle,\rangle$.

Definition 1.2. Suppose (π,V) is a representation of a group G_1 on a finite-dimensional complex vector space V. An *invariant Hermitian form* on V is a Hermitian pairing

$$\langle\cdot,\cdot\rangle\colon V\times V\to\mathbb{C}$$

with the property that

$$\langle\pi(g)v,\pi(g)w\rangle=\langle v,w\rangle\qquad(v,w\in V,\ g\in G_1).$$

The representation π is *Hermitian* if it is endowed with a nondegenerate invariant Hermitian form, and *unitary* if in addition this form is positive definite.

If G_1 is a connected real Lie group with Lie algebra $\mathfrak{g}_1$, then π is determined by its differential (still called π)

$$\pi\colon \mathfrak{g}_1 \to \mathrm{End}(V),$$

a Lie algebra representation. The condition for the Hermitian form to be invariant is equivalent to

$$\langle \pi(X)v, w\rangle + \langle v, \pi(X)w\rangle = 0 \qquad (v, w \in V,\ X \in \mathfrak{g}_1);$$

that is, that the real Lie algebra $\mathfrak{g}_1$ acts by skew-Hermitian operators.

A Hermitian form on a finite-dimensional vector space V has a *signature* which for us will be a triple $(p, q, z) \in \mathbb{N}^3$: here p is the dimension of a maximal positive-definite subspace of V, q is the dimension of a maximal negative-definite subspace, and z is the dimension of the radical. Sylvester's law of inertia says that p, q, and z are well-defined, and that

$$p + q + z = \dim(V). \tag{1}$$

Proposition 1.3 (Schur's Lemma). *Suppose* $(\pi, V) \in (\widehat{G_1})_{\mathit{fin}}$ *(notation* (4)*). Then any two nonzero invariant Hermitian forms on* V *are nondegenerate, and differ by a real nonzero scalar. In particular, the signature* $(p(\pi), q(\pi))$ *is well-defined up to interchanging* p *and* q*.*

Here is an outline of the algorithm in [4] for determining the unitary irreducible representations of a real reductive group.

Algorithm 1. Suppose G_1 is the group of real points of a complex connected reductive algebraic group.

1. *List* all the irreducible representations of G_1 admitting a nonzero invariant Hermitian form.
2. For each such irreducible π, *choose* a nonzero invariant form $\langle , \rangle_\pi$.
3. For each form $\langle , \rangle_\pi$, *calculate* the signature $(p(\pi), q(\pi))$.
4. *Check* whether one of $p(\pi)$ and $q(\pi)$ is zero; in this case, π is an irreducible unitary representation.

We have explained this algorithm in the case of finite-dimensional representations. For infinite-dimensional representations step 1 is the Langlands classification, and what it means to calculate the signature of an invariant form on an infinite-dimensional representation is discussed in [4].

Of these steps, (1) was carried out by Knapp and Zuckerman about 1976; there is an account in [7, Chapter 16]. Their argument was a reduction of the problem to the special case of *representations with real infinitesimal character*. We will not recall the precise definition (see [4, Definition 5.5] or [14, Definition 5.4.11]). The nature of the reduction provided at the same time a reduction of (2)–(4): the entire problem of understanding unitary irreducible representations was reduced to the case of real infinitesimal character. We will therefore concentrate henceforth on this case. (If G_1 is real semisimple, then every finite-dimensional representation of G_1 has real infinitesimal character; so the reduction is invisible on the level of finite-dimensional representations.)

Before we look at an example, one more general idea is useful. A fundamental idea in the representation theory of a real Lie group (or Lie algebra) is to *complexify* the group (or Lie algebra), and take advantage of the (simpler and stronger) structural results available for complex Lie algebras and groups. This is particularly easy for Lie algebras: any real Lie algebra $\mathfrak{g}_1$ has a natural complexification

$$\mathfrak{g} =_{\text{def}} \mathfrak{g}_1 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_1 \oplus i\mathfrak{g}_1. \tag{2a}$$

(The distinction between $\mathfrak{g}_1$ and $\mathfrak{g}$ in this notation seems a little obscure and hard to remember. In the body of the paper, G_1 will usually be something like $G(\mathbb{R})$, and $\mathfrak{g}_1$ will be $\mathfrak{g}(\mathbb{R})$.) The extra structure on the complex Lie algebra $\mathfrak{g}$ that remembers $\mathfrak{g}_1$ is a *real form*: a conjugate-linear real Lie algebra automorphism of order two

$$\sigma_1 \colon \mathfrak{g} \to \mathfrak{g}, \qquad \sigma_1(X + iY) = X - iY \quad (X, Y \in \mathfrak{g}_1). \tag{2b}$$

Any real Lie algebra representation $\pi_{\mathbb{R}}$ of $\mathfrak{g}_1$ on a complex vector space V gives rise to a *complex* Lie algebra representation

$$\pi_{\mathbb{C}}(X + iY) = \pi_{\mathbb{R}}(X) + i\pi_{\mathbb{R}}(Y) \qquad (X, Y \in \mathfrak{g}_1); \tag{2c}$$

and of course $\pi_{\mathbb{R}}$ can be recovered from $\pi_{\mathbb{C}}$ by restriction. This is so elementary and fundamental that it usually goes unsaid, and the subscripts $\mathbb{R}$ and $\mathbb{C}$ on π are not used.

The reason we make this explicit now is that an invariant Hermitian form for $\pi_{\mathbb{R}}$ is almost *never* invariant for $\pi_{\mathbb{C}}$; if $\langle \pi_{\mathbb{R}}(X)v, w\rangle + \langle v, \pi_{\mathbb{R}}(X)w\rangle = 0$ $(x \in \mathfrak{g}_1)$, then

$$\langle \pi_{\mathbb{C}}(iX)v, w\rangle + \langle v, \pi_{\mathbb{C}}(iX)w\rangle = i(\langle \pi_{\mathbb{R}}(X)v, w\rangle - \langle v, \pi_{\mathbb{R}}(X)w\rangle)$$

and there is no reason for this to be 0. What is true is that *the Hermitian form $\langle \cdot, \cdot \rangle$ on V is $\pi_{\mathbb{R}}$-invariant (see 1.2) if and only if*

$$\langle \pi_{\mathbb{C}}(Z)v, w\rangle + \langle v, \pi_{\mathbb{C}}(\sigma_1(Z))w\rangle = 0 \qquad (v, w \in V,\ Z \in \mathfrak{g}). \tag{2d}$$

That is, we require that $\pi_{\mathbb{C}}$ *should carry the complex conjugation on $\mathfrak{g}$ to minus Hermitian transpose on operators*. In this case we call $\langle \cdot, \cdot \rangle$ a σ_1*-invariant form* for the representation $\pi_{\mathbb{C}}$ of $\mathfrak{g}$.

The point to remember is that the definition of invariant Hermitian form on a complex representation of a complex Lie algebra $\mathfrak{g}$ *requires* a choice of real form on $\mathfrak{g}$. Changing the real form changes everything: whether an invariant form exists, and what its signature is.

Example 1.4. Suppose $G_1 = SL(3,\mathbb{R})$. The finite-dimensional representations of G_1 are precisely those of the complex Lie algebra $\mathfrak{sl}(3,\mathbb{C})$. The corresponding real form of $\mathfrak{sl}(3,\mathbb{C})$ is

$$\sigma_1(Z) = \overline{Z} \qquad (Z \in \mathfrak{sl}(3,\mathbb{C})),$$

complex conjugation of matrices.

Irreducible finite-dimensional representations of $\mathfrak{sl}(3,\mathbb{C})$ are indexed by highest weights

$$\lambda = (\lambda_1, \lambda_2, \lambda_3), \quad \lambda_1 + \lambda_2 + \lambda_3 = 0, \quad \lambda_p - \lambda_q \in \mathbb{Z}, \quad \lambda_1 \geq \lambda_2 \geq \lambda_3.$$

For example, $E_{(2/3,-1/3,-1/3)}$ is the defining representation on $\mathbb{C}^3$, and $E_{(1,0,-1)}$ is the 8-dimensional adjoint representation. It turns out that the only representations with a nonzero invariant σ_1-invariant Hermitian form are the "Cartan powers of the adjoint representation":

$$E_{(m,0,-m)} = \text{irreducible representation of dimension } (m+1)^3.$$

(This follows from the Knapp–Zuckerman result explained in [7, Chapter 16], but for finite-dimensional representations is probably much older.)

We would like to understand σ_1-invariant Hermitian forms on $E_{(m,0,-m)}$. According to the program described after Proposition 1.3, we need first to *choose* one of the two possible forms. For this (and for much more!) we will use the restriction of representations of G_1 to the maximal compact subgroup

$$K_1 = SO(3).$$

Because each irreducible representation of a compact group has a positive-definite invariant Hermitian form, the positive and negative parts of an invariant form for G_1 may be understood not just as vector spaces (with dimensions) but as representations of K_1 (sums of irreducible representations with multiplicity). It turns out that $E_{(m,0,-m)}$ contains *either* the trivial representation F_1 of K_1 (if m is even), *or* the tautological three-dimensional representation F_3 (if m is odd), but not both. This representation appears with multiplicity one, so any invariant Hermitian form is either positive or negative definite on the subspace F_1 or F_3. We fix our choice of σ_1-invariant form on $E_{(m,0,-m)}$ by requiring

$$\text{form is } \textit{positive} \text{ on } F_1 \text{ and } \textit{negative} \text{ on } F_3.$$

For example, the adjoint representation $E_{(1,0,-1)} \simeq \mathfrak{g}$ has a Cartan decomposition (more precisely, the complexification of the Cartan decomposition of $\mathfrak{g}_1$)

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = F_3 \oplus F_5$$

(skew-symmetric and symmetric traceless matrices), the sum of irreducible representations of K_1 of dimensions 3 and 5. We can choose for our invariant Hermitian form the trace form

$$\langle X, Y\rangle = \mathrm{tr}(XY^*) \qquad (X, Y \in \mathfrak{sl}(3,\mathbb{C})).$$

This form is easily seen to be positive definite on the space of real symmetric matrices (since these have real eigenvalues), and negative definite on the real skew-symmetric matrices (since these have purely imaginary eigenvalues). In particular, it is negative on F_3. In this way we see that the form on $E_{(1,0,-1)}$ has signature $(5,3)$; even better, the signature is (F_5, F_3) as a representation of $SO(3)$.

Here are a few more signatures. We are for the moment simply claiming that these formulas are correct, not explaining where they come from. Always we write F_{2k+1} for the unique irreducible representation of $SO(3)$ of dimension $2k+1$, endowed with a positive-definite invariant Hermitian form.

$$\text{signature of } E_{(3,0,-3)} = ([F_{13} + F_9 + F_7] + F_5, [F_{11} + F_9 + F_7] + F_3);$$

that is,

$$\mathrm{sig}(E_{(3,0,-3)}) = ([F_{13} + F_9 + F_7], [F_{11} + F_9 + F_7]) + \mathrm{sig}(E_{(1,0,-1)}).$$

Similarly we can compute

$$\begin{aligned}\mathrm{sig}(E_{(5,0,-5)}) = ([&F_{21} + F_{17} + F_{15} + F_{13} + F_{11}],\\ &[F_{19} + F_{17} + F_{15} + F_{13} + F_{11}]) + \mathrm{sig}(E_{(3,0,-3)}).\end{aligned}$$

At this point the pattern may be evident: we get the signature for $E_{(2m+1,0,-2m-1)}$ from that for $E_{(2m-1,0,-2m+1)}$ by adding to the positive and negative parts sums of $2m+1$ irreducible representations of K_1. The two added strings are identical except for the first terms, which differ in dimension by two. (The pattern applies even to getting the signature of $E_{(1,0,-1)}$ from that of the (zero) representation $E_{(-1,0,1)}$.)

In the same way, it turns out that

$$\begin{aligned}\mathrm{sig}(E_{(0,0,0)}) &= (F_1, 0)\\ \mathrm{sig}(E_{(2,0,-2)}) &= ([F_9 + F_5], [F_7 + F_5]) + \mathrm{sig}(E_{(0,0,0)})\\ \mathrm{sig}(E_{(4,0,-4)}) &= ([F_{17} + F_{13} + F_{11} + F_9],\\ &\qquad [F_{15} + F_{13} + F_{11} + F_9]) + \mathrm{sig}(E_{(2,0,-2)}).\end{aligned}$$

The pattern is essentially the same as in the odd case: we get the signature for $E_{(2m+2,0,-2m-2)}$ from that for $E_{(2m,0,-2m)}$ by adding to the positive and negative parts sums of $2m+2$ irreducible representations of K_1. The two added strings are identical except for the first terms, which differ in dimension by two.

As a consequence of this inductive description of the signature as a representation of K_1, or more directly, one can show that

$$\mathrm{sig}(E_{(m,0,-m)}) = ([(m+1)^3 + (m+1)]/2, [(m+1)^3 - (m+1)]/2).$$

In particular, $E_{(m,0,-m)}$ is unitary if and only if $m = 0$: the trivial representation is the only finite-dimensional unitary representation of $SL(3,\mathbb{R})$. (The last statement of course has many extremely short proofs; the point of explaining this long argument is that the ideas apply to infinite-dimensional representations of general real reductive groups.)

This is the shape of the calculation made possible by [4]: we find enormous detail about the precise signatures of invariant Hermitian forms, and then (for the purposes of questions about unitarity) throw almost all of this information away. Of course we would be very happy to learn what interesting questions this discarded information is actually addressing.

We now describe how the calculation of signatures is related to a more classical representation-theoretic problem of Clifford theory: how to extend an irreducible representation of a normal subgroup.

We do not wish to use all of the somewhat complicated and delicate hypotheses under which we finally work (involving real algebraic groups and L-groups just as a point of departure). On the other hand, we would like to use notation that is close to being consistent with that of the body of the paper. Here is a compromise.

A key object to consider will be a group extension

$$1 \to G \to {}^{\mathrm{ex}}G \to \{1, \delta\} \to 1, \tag{3a}$$

which we call an *extended group for* G. In the body of the paper, G will very often be a complex connected reductive algebraic group. Perhaps the most familar example of such an extension, and one that we will certainly use (often behind the scenes), is Langlands L-group (12h); there the role of G is played by a (complex connected reductive algebraic) dual group, and $\{1, \delta\}$ is the Galois group of $\mathbb{C}/\mathbb{R}$.

Here is a concrete way to construct such a group extension. Begin with an automorphism $\theta \in \mathrm{Aut}(G)$, with the property that θ^2 is inner:

$$\theta^2 = \mathrm{Int}(g_0) \qquad (g \in G). \tag{3b}$$

(Only the coset $g_0 Z(G)$ is determined by θ.) Then we can define ${}^{\mathrm{ex}}G$ by generators and relations, as the group generated by G and a single additional element h_0, satisfying

$$h_0^2 = g_0, \qquad h_0 g h_0^{-1} = \theta(g) \quad (g \in G). \tag{3c}$$

(This presentation *does* depend on the choice of representative g_0 for the coset $g_0 Z(G)$.) Two automorphisms θ and θ' of G are said to be *inner* to each other if $\theta' \circ \theta^{-1}$ is an inner automorphism. Now it is clear that

$$\{\mathrm{Int}(h)|_G \mid h \in {}^{\mathrm{ex}}G - G\} \quad \text{is an inner class in } \mathrm{Aut}(G). \tag{3d}$$

This is how we will use extended groups: as a place to keep track of and compare representatives of various automorphisms of G.

An *extended subgroup* of ${}^{\mathrm{ex}}G$ is a subgroup ${}^{\mathrm{ex}}G_1$ mapping surjectively to $\{1, \delta\}$. In this case we define $G_1 = G \cap {}^{\mathrm{ex}}G_1$, so that

$$1 \to G_1 \to {}^{\mathrm{ex}}G_1 \to \{1, \delta\} \to 1. \tag{3e}$$

Often we will consider several such subgroups ${}^{\mathrm{ex}}G_1$, ${}^{\mathrm{ex}}G_2$, and so on.

In the body of the paper, the (complex) group G will be a very useful tool, but we will be interested actually in representations only of a real form $G(\mathbb{R})$; then this will be a typical G_1. A little more precisely, if ${}^{\mathrm{ex}}G$ is a complex Lie group, then a *real form* means an antiholomorphic automorphism of order two of real extended Lie groups

$$\sigma_1 \colon {}^{\mathrm{ex}}G \to {}^{\mathrm{ex}}G, \quad \sigma_1(G) = G, \qquad {}^{\mathrm{ex}}G_1 =_{\mathrm{def}} [{}^{\mathrm{ex}}G]^{\sigma_1}. \tag{3f}$$

("Antiholomorphic" means that if f is a local holomorphic function on ${}^{\mathrm{ex}}G$, then $\overline{f \circ \sigma_1}$ is holomorphic as well. This implies in particular that the differential of σ_1 (still denoted σ_1) is a real form of $\mathfrak{g}$ in the sense of (2b)).

Our main results are about the problems of Clifford theory (Proposition 1.5): the relationship between representations of G and of ${}^{\mathrm{ex}}G$. Here is a classical statement of Clifford theory for finite-dimensional representations; in the world of Harish-Chandra modules, the extension to infinite-dimensional representations is easy. Write

$$(\widehat{G})_{\mathrm{fin}} = \text{equiv. classes of fin.-diml. irreducible representations.} \tag{4}$$

Proposition 1.5 (Clifford). *Suppose $G \subset {}^{ex}G$ is an extended group* (3a).

1. *The quotient ${}^{ex}G/G = \{1, \delta\}$ acts on $\widehat{G}_{fin}$, by*

   $$\delta \cdot (\pi, E) = (\pi^h, E), \qquad \pi^h(g) = \pi(hgh^{-1});$$

   here h is any fixed element of ${}^{ex}G - G$ (the non-identity coset of G in ${}^{ex}G$). The equivalence class of π^h is independent of the choice of h.
2. *Define*

   $$\epsilon \colon {}^{ex}G \to \{\pm 1\}, \quad \epsilon(h) = \begin{cases} 1 & (h \in G) \\ -1 & (h \notin G). \end{cases}$$

   Then the group of characters $\{1, \epsilon\}$ of ${}^{ex}G/G$ acts on $(\widehat{{}^{ex}G})_{fin}$ by

   $$\epsilon \cdot \Pi = \Pi \otimes \epsilon.$$

3. *Because these are actions of two-element groups, we have*

   $\pi \in \widehat{G}_{fin}$ **either** *is fixed by δ,* **or** *has a two-element orbit $\{\pi, \delta \cdot \pi\}$.*

   Similarly,

   $\Pi \in (\widehat{{}^{ex}G})_{fin}$ **either** *has a two-element orbit $\{\Pi, \epsilon \cdot \Pi\}$,* **or** *is fixed by ϵ.*

4. *The two-element orbits of δ on $\widehat{G}_{fin}$ are in one-to-one correspondence (by induction from G to ${}^{ex}G$) with the ϵ-fixed elements of $(\widehat{{}^{ex}G})_{fin}$. These are the rep-*

resentations Π of ${}^{ex}G$ whose characters vanish on ${}^{ex}G - G$; their characters on G are the sum of the two corresponding characters of G.

5. *The two-element orbits of ϵ on $(\widehat{{}^{ex}G})_{fin}$ are in one-to-one correspondence (by restriction to G) with the δ-fixed elements of $\widehat{G}_{fin}$. The two extensions Π and Π' of such a π have characters on ${}^{ex}G - G$ differing by sign; their characters on G agree with that of π.*
6. *Suppose (π, E) is a δ-fixed element of $\widehat{G}_{fin}$. Then an extension of π to ${}^{ex}G$ may be constructed as follows. Fix any element $h_0 \in {}^{ex}G - G$. Let A_π be a nonzero intertwining operator from π to π^h:*

$$A_\pi \pi(g) = \pi(h_0 g h_0^{-1}) A_\pi.$$

This requirement determines A_π up to a multiplicative scalar. Write $g_0 = h_0^2 \in G$. After modifying A_π by an appropriate scalar, we may arrange

$$A_\pi^2 = \pi(g_0);$$

with this additional condition, A_π is determined up to multiplication by ± 1. Each choice of A_π determines an extension Π of π, by the requirement

$$\Pi(h_0) = A_\pi.$$

Suppose we understand the character theory of the smaller group G. Because of this proposition, in order to understand the character theory of ${}^{\mathrm{ex}}G$, we must understand, for each δ-fixed irreducible representation of G, the character of *some extension* of it on ${}^{\mathrm{ex}}G - G$. There are always exactly two such extensions, whose characters on ${}^{\mathrm{ex}}G - G$ differ by sign; our task will be to find a way (for the particular groups of interest) to specify one of these two extensions.

Suppose now (as we will for the body of this paper) that G is a complex connected reductive algebraic group, and that

$$\sigma\colon G \to G \tag{5a}$$

is a real form: an antiholomorphic automorphism of order two of real Lie groups. The corresponding real reductive algebraic group is

$$G(\mathbb{R}, \sigma) = G(\mathbb{R}) =_{\text{def}} G^\sigma, \tag{5b}$$

a real Lie group with Lie algebra

$$\mathfrak{g}(\mathbb{R}) =_{\text{def}} \mathfrak{g}^\sigma. \tag{5c}$$

Now G has one particularly interesting (conjugacy class of) real form(s), the *compact real form σ_c*. It is characterized up to conjugation by G by the requirement that

$$G(\mathbb{R}, \sigma_c) \text{ is compact.} \tag{5d}$$

Elie Cartan showed that σ_c may be chosen to *commute* with the real form σ, and that this requirement determines σ_c up to conjugation by $G(\mathbb{R},\sigma)$. Because of the commutativity, the composition

$$\theta = \sigma \circ \sigma_c = \sigma_c \circ \sigma \tag{5e}$$

is an algebraic involution of G of order two called the *Cartan involution*; it is determined by σ up to conjugation by $G(\mathbb{R},\sigma)$. The group of fixed points

$$K = G^\theta \tag{5f}$$

is a (possibly disconnected) complex reductive algebraic subgroup of G. The two real forms σ and σ_c of G both preserve K, and act the same way there; the corresponding real form

$$K(\mathbb{R}) = G(\mathbb{R},\sigma) \cap K = G(\mathbb{R},\sigma) \cap G(\mathbb{R},\sigma_c) = G(\mathbb{R},\sigma_c) \cap K \tag{5g}$$

is a maximal compact subgroup of $G(\mathbb{R},\sigma)$ and a maximal compact subgroup (the compact real form) of K.

We wish to understand σ-invariant Hermitian forms on representations of $G(\mathbb{R})$. The next proposition recalls the classical solution (by Cartan and Weyl) of a related problem, and then relates the two problems.

Proposition 1.6. *Suppose we are in the setting* (5).

1. *Finite-dimensional algebraic representations of K may be identified with finite-dimensional continuous representations of the compact real form $K(\mathbb{R})$.*
2. *Finite-dimensional algebraic representations of G may be identified with finite-dimensional continuous representations of the compact real form $G(\mathbb{R},\sigma_c)$.*
3. *Every finite-dimensional irreducible algebraic representation (π,E) of G admits a positive-definite σ_c-invariant Hermitian form $\langle\cdot,\cdot\rangle_c$, unique up to positive scalar multiple:*
$$\langle \pi(g)v, w\rangle_c = \langle v, \pi(\sigma_c(g))^{-1}w\rangle_c.$$
4. *The finite-dimensional irreducible algebraic representation (π,E) of G admits a σ-invariant Hermitian form $\langle\cdot,\cdot\rangle$ if and only if there is a nonzero linear operator, self-adjoint with respect to the form $\langle,\rangle_c$,*
$$A_\pi \colon E \to E, \qquad A_\pi^* = A_\pi$$
with the property that
$$A_\pi \pi(g) = \pi(\theta(g))A_\pi \qquad (g \in G).$$
In particular, A_π commutes with the action of K. These requirements determine A_π up to a real multiplicative scalar. In this case the σ-invariant form is
$$\langle v, w\rangle = \langle v, A_\pi w\rangle_c.$$

5. *In the setting of (4),* A_π^2 *must commute with the action of* π*, and so must be a nonzero scalar. Because* A_π *is self-adjoint with respect to the positive Hermitian form* $\langle , \rangle_c$*, the scalar is necessarily positive real:*

$$A_\pi^2 = r_\pi I_E, \qquad r_\pi \in \mathbb{R}^{+,\times}.$$

6. *The signature of this* σ*-invariant form is*

$$\mathrm{sig}(E) = (+1 \text{ eigenspace of } (r_\pi^{-1/2})A_\pi)), -1 \text{ eigenspace of } (r_\pi^{-1/2})A_\pi);$$

here the positive and negative parts are representations of K.

The conditions on A_π in Proposition 1.6 look like conditions in Proposition 1.5 for defining a representation of an extended group. We deduce easily

Corollary 1.7. *In the setting* (5)*, suppose also that* G *is part of an extended group as in* (3)*; and that*

$$\theta = \mathrm{Int}(\xi), \quad \xi \in {}^{ex}G - G, \quad \xi^2 = z \in Z(G).$$

Then a finite-dimensional irreducible algebraic representation (π, E) *of* G *admits a* σ*-invariant Hermitian form if and only if* π *has an extension* Π *to* ${}^{ex}G$*. In that case define a nonzero complex scalar* z_π *so that*

$$\pi(z) = z_\pi I_E,$$

and choose a square root ω_π *of* z_π*. Then the* σ*-invariant Hermitian form on* E *may be taken to be*

$$\langle v, w\rangle = \langle v, \omega_\pi^{-1}\Pi(\xi)w\rangle_c.$$

The signature of this σ*-invariant form is*

$$\mathrm{sig}(E) = (+1 \text{ eigenspace of } \omega_\pi^{-1}\Pi(\xi), -1 \text{ eigenspace of } \omega_\pi^{-1}\Pi(\xi)).$$

In particular, the difference between the dimensions of the positive and negative parts is equal to ω_π^{-1} *times the character value* $\mathrm{tr}(\Pi(\xi))$.

There is no difficulty in finding the extended group needed in this corollary: one can use for example

$$^{\mathrm{ex}}G = G \rtimes \{1, \xi\}, \tag{6}$$

with ξ acting on G by the Cartan involution θ. In this case $z = 1$, so the statement of the corollary simplifies a bit. We allow for more general extended groups because those will turn out to be useful for the bookkeeping we want to do.

Corollary 1.7 says that understanding the existence and signatures of σ-invariant Hermitian forms on algebraic representations of G is equivalent to understanding the algebraic representations of the disconnected (complex reductive algebraic) group ${}^{\mathrm{ex}}G$. Here is what happens in the case of $SL(3)$.

Proposition 1.8. *Suppose $G = SL(3)$, with the real form $G(\mathbb{R},\sigma) = SL(3,\mathbb{R})$ given by complex conjugation of matrices. Then a compact real form of $SL(3)$ is $SU(3)$, with complex conjugation given by inverse Hermitian transpose:*

$$\sigma_c(g) = {}^t\overline{g}^{-1} \qquad (g \in SL(3,\mathbb{C})).$$

Then σ_c and σ commute, so the Cartan involution is

$$\theta(g) = {}^tg^{-1}, \quad K = SO(3,\mathbb{C}), \quad K(\mathbb{R}) = SO(3).$$

Twisting by θ carries the representation of highest weight $(\lambda_1,\lambda_2,\lambda_3)$ (see Example 1.4) to the one of highest weight $(-\lambda_3,-\lambda_2,-\lambda_1)$. In particular, the only representations fixed are the various $(\pi_m, E_{(m,0,-m)})$.

For such a representation, we can therefore find an operator

$$A_m \colon E_{(m,0,-m)} \to E_{(m,0,-m)}, \quad A_m\pi_m(g) = \pi_m({}^tg^{-1}) \quad (g \in SL(3)).$$

This requirement specifies A_m up to a scalar; we can specify it precisely by requiring that A_m act by $+1$ on the unique largest $SO(3)$ representation F_{4m+1} inside $E_{(m,0,-m)}$.

With this choice, we can extend π_m to a representation Π_m of the disconnected group ${}^{ex}SL(3)$ of (6), by defining

$$\Pi_m(\xi) = A_m.$$

The semisimple conjugacy classes of $SL(3)$ on the non-identity component of ${}^{ex}SL(3)$ are represented by

$$\left\{ h(z) = \begin{pmatrix} 0 & 0 & z \\ 0 & 1 & 0 \\ -z^{-1} & 0 & 0 \end{pmatrix} \xi \right\};$$

here $h(z)$ is conjugate to $h(z^{-1})$. By a version of the Weyl character formula (for example [16, Theorem 1.43]), the trace of this element in the extended representation Π_m is

$$\mathrm{tr}(\Pi_m(h(z))) = (-1)^m(z^{2m+2} - z^{-2m-2})/(z^2 - z^{-2})$$

for z not a fourth root of 1. The element ξ is conjugate to $h(\pm i)$; so by L'Hôpital's rule,

$$\mathrm{tr}(\Pi_m(\xi)) = \mathrm{tr}(\Pi_m(h(i))) = m+1.$$

The difference between the positive and negative parts of the signature of the σ-invariant Hermitian form defined by Π_m is therefore $m+1$, so this is the same form described in Example 1.4.

Perhaps the most challenging part of proving this proposition is to verify that ξ is conjugate to $h(\pm i)$, but this can be done.

Kazhdan–Lusztig theory for computing irreducible characters typically takes place in a free $\mathbb{Z}[q]$-module with basis indexed by the irreducible characters of interest. This $\mathbb{Z}[q]$ modules carries a representation of the Hecke algebra, and the Kazhdan–Lusztig polynomials are determined by the Hecke module structure.

In something like the setting of Proposition 1.5 (where we already understand characters on G, and so wish to understand just characters on ${}^{\text{ex}}G - G$) this suggests that we will be interested in a free $\mathbb{Z}[q]$-module having a basis $\{m_\Pi\}$ indexed by *one* irreducible representation Π from each pair $\Pi \neq \Pi' = \Pi \otimes \epsilon$. In this module, we will think of

$$m_{\Pi'} = -m_\Pi$$

(corresponding to the fact that the characters of Π and Π' sum to zero on ${}^{\text{ex}}G - G$). All of this is explained more precisely in Section 7.

The computational problem in implementing the Kazhdan–Lusztig algorithm is that we know precisely how to parametrize the δ-fixed irreducibles π of the smaller group G; but a δ-fixed irreducible corresponds only to a pair $\{\Pi, \Pi'\}$, and so only to a basis vector of the Hecke module *defined up to sign*. We need an equally precise parametrization of irreducibles of ${}^{\text{ex}}G$; that is, of how to specify one of the two possible extensions of π to ${}^{\text{ex}}G$. In Proposition 1.8 this happened with the requirement that A_m act by $+1$ on F_{4m+1}. This amounts to a condition involving the action of a particular element of the larger group ${}^{\text{ex}}G$ on a highest weight vector.

Corollary 1.7 shows that (in the setting (5)) understanding σ-invariant Hermitian forms on finite-dimensional representations is closely related to understanding the extensions to ${}^{\text{ex}}G(\mathbb{R},\sigma)$ of irreducible representations of $G(\mathbb{R},\sigma)$.

An important special case is when ξ of (6) acts by an inner involution of G. In this case write $\xi(g) = xgx^{-1}$ for some $x \in G$. Then the map $\xi \to (x,\epsilon)$ induces an isomorphism

$$ {}^{\text{ex}}G = G \rtimes \{1,\xi\} \simeq G \times \mathbb{Z}/2\mathbb{Z} = G \times \{1,\epsilon\}. \tag{7}$$

In this, the *equal rank case*, there is no essential new information in the representation theory of ${}^{\text{ex}}G$, and it is enough to work with G itself.

With the appropriate generalizations, this can be made to work for infinite-dimensional representations as well. This is discussed in detail in [4]. Just as in the case of finite-dimensional representations, it is not necesary to use the extended group in the case of an equal rank group. See [4, Section 11].

In the unequal rank case, this requires (at least implicitly) understanding the analogues of highest weights—Lie algebra cohomology for maximal nilpotent subalgebras $\mathfrak{n}$—by which infinite-dimensional representations (π, E) of real reductive groups are classified. A little more precisely, one looks at the normalizer $G_\mathfrak{n}$ of $\mathfrak{n}$ in G. This group acts by a character χ_π on a Lie algebra cohomology space $H^*(\mathfrak{n}, E)$, and the character χ_π determines the representation π. To specify an extension (Π, E) of π to ${}^{\text{ex}}G$, one needs an extension χ_Π of χ_π to the normalizer ${}^{\text{ex}}G_\mathfrak{n}$ of $\mathfrak{n}$ in ${}^{\text{ex}}G$. To get that, we can fix any element

$$h_\mathfrak{n} \in {}^{\text{ex}}G_\mathfrak{n} - G_\mathfrak{n}. \tag{8a}$$

Necessarily

$$h_{\mathfrak{n}}^2 = g_{\mathfrak{n}} \in G_{\mathfrak{n}}. \tag{8b}$$

An extension Π of of π to ${}^{ex}G$ is specified by specifying the single character value $\chi_\Pi(h_{\mathfrak{n}})$, which may be either square root of $\chi_\pi(g_{\mathfrak{n}})$:

$$\text{extension } \Pi \text{ of } \pi \quad \longleftrightarrow \quad \text{square root } \chi_\Pi(h_{\mathfrak{n}}) \text{ of } \chi_\pi(g_{\mathfrak{n}}). \tag{8c}$$

What makes matters difficult is that the cohomology classes needed for different representations involve different maximal nilpotent subalgebras, and (as it turns out) necessarily different elements $h_{\mathfrak{n}}$. Even worse, for a single $\mathfrak{n}$, there may be no preferred choice of $h_{\mathfrak{n}}$. We need to have a way to keep track of choices of these elements $h_{\mathfrak{n}}$, and of the square roots $\chi_\Pi(h_{\mathfrak{n}})$.

A natural way to reduce choices would be to try to arrange for $h_{\mathfrak{n}}$ to have order two; in that case $g_{\mathfrak{n}} = 1$, so $\chi_\pi(g_{\mathfrak{n}}) = 1$, and the choice $\chi_\Pi(h_{\mathfrak{n}})$ must be ± 1. This is more or less what happened in Proposition 1.8, and we were then able to make the "natural" choice $\chi_\Pi(h_{\mathfrak{n}}) = 1$. But in general we cannot always arrange for $h_{\mathfrak{n}}$ to have order 2. It turns out that there is behavior like the example of $G = \mathbb{Z}/4\mathbb{Z} = \{\pm 1, \pm i\}$ sitting inside the quaternion group ${}^{ex}G$ of order 8: every element $\{\pm j, \pm k\}$ of the non-identity coset has order exactly 4. Once we are forced to consider a case when $\chi_\pi(g_{\mathfrak{n}}) = -1$, it is easy to believe that there can be no preferred choice of square root.

This gives a hint at the difficulties we face. To explain in more detail their resolution, we begin with the extension of the Cartan–Weyl highest weight theory to parametrize representations. This is provided by the Langlands classification, which is phrased in terms of the complex reductive dual group. Langlands' results in their original form parametrize not individual representations but "L-packets," which are collections of finite sets of irreducible representations for each of several different real forms of G. To use the construction of (6) would require introducing a different extended group for each of these different real forms. This is inconvenient at best, and is inconsistent with the cleanest formulation of the Langlands classification.

A glimpse of this inconvenience is the description of conjugacy classes in the extended group given in Proposition 1.8. What is good about the elements $h(z)$ defined there is that they normalize the standard Borel subgroup (consisting of upper triangular matrices) in $SL(3)$; the element ξ does not. It is this good property that allows one to write a nice Weyl character formula for the elements $h(z)$. We recall next the notion of *pinning* for a reductive algebraic group, and the derived notion of *distinguished automorphism*; these are required for the formulation of the Langlands classification made in Section 3.

Definition 1.9. Suppose G is a complex connected reductive algebraic group. A *pinning* of G consists of

1. a Borel subgroup $B \subset G$;
2. a maximal torus $H \subset B$; and
3. for each simple root α, a choice of basis vector $X_\alpha \in \mathfrak{g}_\alpha$.

The pair $H \subset B$ is determined by $\{X_\alpha\}$, so we can just write $(G, \{X_\alpha\})$ for the pinning.

An algebraic automorphism δ_0 of G is called *distinguished* (with respect to this pinning) if the differential of δ_0 permutes the chosen simple root vectors X_α. (As a consequence, δ_0 must preserve H and B.)

If θ_0 is a distinguished automorphism of order one or two, we define the *distinguished extended group* to be the algebraic group ${}^\Gamma G$ generated by G and one more element ξ_0, subject to the relations

$$\xi_0^2 = 1, \qquad \xi_0 g = \theta_0(g)\xi_0 \quad (\mathfrak{g} \in G).$$

Recall that two automorphisms δ and δ' of G are said to be *inner* to each other if $\delta' \circ \delta^{-1}$ is an inner automorphism.

Proposition 1.10 ([11, Corollary 2.14]). *Suppose $(G, \{X_\alpha\})$ is a complex connected reductive algebraic group with a pinning. Then any automorphism δ of G is inner to a* unique *distinguished automorphism δ_0. Necessarily the order of δ_0 divides the order of δ (where we make the conventions that any nonzero natural number divides infinity, and infinity divides itself). If in addition δ is semisimple (for example, if δ has finite order), then δ is conjugate by G to an automorphism $\mathrm{Ad}(h)\delta_0$, for some (usually not unique) $h \in H$.*

In case θ_0 has order one or two, the proposition says that *every automorphism θ of G inner to θ_0 may be realized by the conjugation action of an element ξ of the nonidentity coset $G\xi_0$ of the corresponding distinguished extended group*. The difference from (6) is that, even if $\theta^2 = 1$, the element ξ^2 may be a *nontrivial* element of $Z(G)$. This turns out to be a small price to pay for having a single extended group to work with (as θ varies over an inner class).

The Cartan involution θ (and therefore the extended group ${}^{\mathrm{ex}}G$) is playing a double role in Corollary 1.7: first, specifying the real form $G(\mathbb{R})$; and second, specifying an automorphism of $G(\mathbb{R})$ by which we wish to twist representations. It will be convenient to separate these two roles: to study the twisting of representations of $G(\mathbb{R})$ by a second automorphism δ.

Section 2 establishes the required notation for "doubly extended groups," and recalls also Langlands' L-group. Section 3 recalls from [3] a formulation of the Langlands classification well-suited to calculation. Section 4 computes the twisting action of δ on representations. The idea here (exactly as in the original work of Knapp and Zuckerman recorded in [7]) is that this is a fairly elementary inspection of the twisting action on *parameters* for representations.

Section 5 describes a way to add information to a δ-fixed parameter—essentially choices of elements $h_{\mathfrak{n}}$ and $\chi_\Pi(h_{\mathfrak{n}})$ discussed in (8)—to specify a representation of the corresponding extended group ${}^\delta G(\mathbb{R})$. Particularly because the extended group element $h_{\mathfrak{n}}$ is not unique, the question of when two of these extended representations are equivalent is a bit subtle; Section 6 answers this question.

In this way we are able to write explicitly a basis (not just a basis defined up to sign) for the Hecke module considered in [10], and the Kazhdan–Lusztig polynomials are determined by this Hecke module. Precise formulas for the action of

Hecke algebra generators on the basis are written in Section 7. Each such formula involves one to four basis vectors in the module. In [10] it was shown that these one to four basis vectors could be chosen so that the action of the generator was given by a specified matrix (of size one to four). A typical example (the *only* example in the original paper [6]) is

$$\begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix}. \tag{9}$$

The technical problem that led to this paper, mentioned at the beginning of the introduction, is that *these nice choices of basis vectors cannot be made consistently as the Hecke algebra generator varies*. The result is that if we fix a single choice of basis for the Hecke module, then the actions of some of the Hecke algebra generators will be given by matrices made of blocks not only like (9), but also by conjugates of such a matrix by a diagonal matrix with entries ± 1. A typical example is

$$\begin{pmatrix} 0 & -q \\ -1 & q-1 \end{pmatrix}. \tag{10}$$

The point of the formulas in Section 7 is to say precisely where the minus signs must go. In order to do this, one needs to say how to manipulate our extended parameters to get the nice basis vectors discussed in [10]. There are two cases where this manipulation is somewhat more complicated, and they are described in detail in Sections 8 and 9. Ultimately this gives an explicit algorithm for computing the polynomials of [10], which is being implemented in the `atlas of Lie groups and Representations` software [17]. The application to our computation of Hermitian forms is [4, Theorem 19.4].

A guiding principle in formulating these results is the fundamental duality theorem originating in [6, Theorem 3.1], and extended to Harish-Chandra modules in [15]. Section 11 describes how to prove this for the Hecke modules in the twisted setting. The heart of the proof in every case is that a "transpose" of one Hecke algebra action is equal to another Hecke algebra action; explicitly, that the transpose of the matrix giving an action of a generator is equal to the matrix giving the action of the same generator on a different module. That such a statement is true up to signs was clear from [10]; with the specification of the signs in this paper we are able to prove it completely.

2 Setting

Our first goal is to understand which representations are fixed by a given outer automorphism, and how to to write down the corresponding representations of the extended group. We begin by setting up some notation in this section, discuss the `atlas` parametrization of representations in Section 3, and the action of twisting on these parameters in Section 4.

We start with a connected complex reductive algebraic group G, equipped with a pinning (Definition 1.9). Acting on this we have two commuting distinguished involutive automorphisms:

$$\xi_0\colon (G,B,H)\to(G,B,H),\qquad \delta_0\colon (G,B,H)\to(G,B,H), \tag{11a}$$

satisfying

$$\xi_0(X_\alpha)=X_{\xi_0(\alpha)},\quad \delta_0(X_\alpha)=X_{\delta_0(\alpha)}\qquad (\alpha,\ \xi(\alpha),\ \delta_0(\alpha)\in\Pi). \tag{11b}$$

See Definition 1.9 and [1, p. 34 or p. 51].

The automorphism ξ_0 defines the inner class of real forms under consideration; it is the unique Cartan involution in the inner class which is distinguished, and is the Cartan involution of the "most compact" real form in the inner class. The automorphism δ_0 defines the twisting of representations that we will consider. Since any automorphism is inner to a distinguished one there is no loss in assuming δ_0 is distinguished.

We will abuse notation and use these automorphisms to define a semidirect product of G with the Klein 4-group $(\mathbb{Z}/2\mathbb{Z})^2$:

$${}^{\Delta}G = G\rtimes\{1,\xi_0,\delta_0,\xi_0\delta_0\}. \tag{11c}$$

The superscript Δ is supposed to suggest "double." The abuse of notation is that from now on ξ_0 may denote an element of ${}^{\Delta}G$ (which by definition is never the identity) *or* an automorphism of G (which is the identity exactly when ξ_0 defines the equal rank inner class).

It is helpful to use also the corresponding *large* ([1, p. 51]) involutive automorphism. As in [1] we write

$$e\colon \mathfrak{h}\to H,\qquad e(X)=\exp(2\pi iX); \tag{11d}$$

this is a surjective homomorphism from the Lie algebra onto H, with kernel equal to $X_*(H)$. Also we write

$$\rho=\frac{1}{2}\sum_{\beta\in R^+(G,H)}\beta,\quad \rho^\vee=\frac{1}{2}\sum_{\beta\in R^+(G,H)}\beta^\vee. \tag{11e}$$

Then $\alpha(e(\rho^\vee/2))=-1$ for every simple root α; so if we define

$$\xi_1=e(\rho^\vee/2)\xi_0\in H\xi_0, \tag{11f}$$

then this element of ${}^{\Delta}G$ acts on G as an involutive automorphism satisfying

$$\xi_1|_H=\xi_0|H,\quad \xi_1(X_\alpha)=-X_{\xi(\alpha)},\qquad (\alpha\in\Pi). \tag{11g}$$

This element satisfies

$$\xi_1^2 = e(\rho^\vee) =_{\text{def}} z(\rho^\vee) \in Z(G), \tag{11h}$$

a central element of order (one or) two.

Our torus $H \subset G$ has a well-defined (that is, uniquely defined up to unique isomorphism) dual torus

$$^\vee H = X^*(H) \otimes_{\mathbb{Z}} \mathbb{C}^\times. \tag{12a}$$

The characters and cocharacters of $^\vee H$ are naturally identified with the cocharacters and characters of H:

$$X^*(^\vee H) \simeq X_*(H), \qquad X_*(^\vee H) \simeq X^*(H). \tag{12b}$$

The isomorphisms here are canonical, and respect the pairings into $\mathbb{Z}$.

The automorphisms ξ_0 and δ_0 of H (cf. (11)) define automorphisms ${}^t\xi_0$ and ${}^t\delta_0$ of $X^*(H)$, and therefore

$$^\vee\xi_0 =_{\text{def}} -w_0{}^t\xi_0, \qquad {}^\vee\delta_0 =_{\text{def}} {}^t\delta_0 \tag{12c}$$

of $X^*(H)$ and of $^\vee H$. Here we write

$$w_0 \in W(G,H) \simeq W(^\vee G, {}^\vee H) \tag{12d}$$

for the unique longest element, which carries $R^+(G,H)$ to $-R^+(G,H)$. *Notice the presence of a minus sign in the definition of* $^\vee\xi_0$ *(partly "corrected" by the factor of* w_0*) and its absence in the definition of* $^\vee\delta_0$. This is the way things are. One way to understand it is that ξ is related to the Cartan involution for G, which is less fundamental and natural than the Galois action for a real form. The Cartan involution acts on the root datum (with respect to a real θ-stable Cartan) by the *negative* of the Galois action on the root datum; and it is this minus sign which accounts for the minus sign in (12c).

Now we construct a dual group $^\vee G \supset {}^\vee H$, whose root datum is dual to that of G:

$$^\vee G \supset {}^\vee B = {}^\vee H {}^\vee N, \quad R^+(^\vee G, {}^\vee H) = \{\beta^\vee \mid \beta \in R^+(G,H)\}. \tag{12e}$$

We choose also a pinning: nonzero root vectors

$$\{X_{\alpha^\vee} \mid \alpha^\vee \in \Pi^\vee\} \subset {}^\vee\mathfrak{n}. \tag{12f}$$

Such a choice of dual group and pinning is unique up to unique isomorphism. Because the automorphisms $^\vee\xi_0$ and $^\vee\delta_0$ respect the based root datum, they extend uniquely to (distinguished) automorphisms

$$\begin{aligned} {}^\vee\xi_0 &\colon (^\vee G, {}^\vee G, {}^\vee H) \to (^\vee G, {}^\vee B, {}^\vee H), \quad {}^\vee\xi_0(X_{\alpha^\vee}) = X_{-w_0\xi_0(\alpha)^\vee} \\ {}^\vee\delta_0 &\colon (^\vee G, {}^\vee B, {}^\vee H) \to (^\vee G, {}^\vee B, {}^\vee H), \quad {}^\vee\delta_0(X_{\alpha^\vee}) = X_{\delta_0(\alpha)^\vee}. \end{aligned} \tag{12g}$$

Automatically ${}^\vee\delta_0$ and ${}^\vee\xi_0$ commute. By definition the *L-group of G* is the semidirect product

$$ {}^LG = {}^\vee G \rtimes \{1, {}^\vee\xi_0\}. \tag{12h} $$

(A little more precisely, it is this group endowed with the ${}^\vee G$-conjugacy class of $({}^\vee B, \{X_{\alpha^\vee}\}, {}^\vee\xi_0)$.)

Just as for G, it is convenient to have in hand also the *large* representative

$$ {}^\vee\xi_1 = e(\rho/2)\xi_0, \qquad {}^\vee\xi_1(X_{\alpha^\vee}) = -X_{-w_0\xi_0(\alpha)^\vee}. \tag{12i} $$

Again this element satisfies

$$ {}^\vee\xi_1^2 = e(\rho) =_{\text{def}} z(\rho) \in Z({}^\vee G), \tag{12j} $$

a central element of order (one or) two.

We say a little more about the identification of Weyl groups in (12d). Define

$$ \begin{aligned} s_\alpha \in \mathrm{Aut}(X_*(H)), \quad s_\alpha(t) = t - \langle \alpha, t\rangle \alpha^\vee \\ W(G,H) = \langle s_\alpha \mid \alpha \in \Pi \rangle \subset \mathrm{Aut}(X_*(H)). \end{aligned} \tag{12k} $$

Then the identification

$$ \mathrm{Aut}(X_*(H)) \supset W(G,H) \simeq W({}^\vee G, {}^\vee H) \subset \mathrm{Aut}(X^*(H)) $$

is given by

$$ s_\alpha \mapsto s_{\alpha^\vee}, \qquad w \mapsto {}^t w^{-1}. \tag{12l} $$

3 Atlas parameters

The basic reference for this section is [3].

As explained after Proposition 1.10, we are going to represent involutive automorphisms of G (briefly, *involutions*) by the conjugation action of elements of $G\xi_0$. For this purpose we introduce the set of *strong involutions*:

$$ \mathcal{I} = \{\xi \in G\xi_0 \mid \xi^2 \in Z(G)\}. \tag{13a} $$

If $\xi \in \mathcal{I}$, then

$$ \theta_\xi = \mathrm{int}(\xi), \qquad K_\xi = G^{\theta_\xi} = \mathrm{Cent}_G(\xi). \tag{13b} $$

is an involutive automorphism of G, in the inner class of ξ_0; and every such involutive automorphism arises this way. We need to allow $\xi^2 \in Z(G)$ (and not merely $\xi^2 = 1$) because not every involution in the inner class of ξ_0 arises from an element ξ of order 2. (But we *can* easily arrange for ξ to have order a power of 2.) The central element

$$ z = \xi^2 \in Z(G) \tag{13c} $$

is called the *central cocharacter* of the strong involution ξ.

A *strong real form* of G is a G-conjugacy class $\mathcal{C} \subset \mathcal{I}$. The central cocharacter is constant on $\mathcal{C}$, so we may write it as

$$z(\mathcal{C}) = \xi^2 \in Z(G) \qquad (\xi \in \mathcal{C}). \tag{13d}$$

The various involutions $\{\theta_\xi \mid \xi \in \mathcal{C}\}$ form a single G-conjugacy class of involutive automorphisms of G, so the subgroups $\{K_\xi \mid \xi \in \mathcal{C}\}$ are a single G-conjugacy class as well. If G is adjoint, then these three G-conjugacy classes (strong involutions, involutions, and fixed point subgroups) are identified by the natural maps

$$\xi \to \theta_\xi \to K_\xi.$$

If G is not adjoint, however, the first of these maps need not be one-to-one: choosing a strong involution is more restrictive than choosing an involution.

Here is the reason that strong involutions and strong real forms are useful.

Proposition 3.1. *Suppose ξ and ξ' are strong involutions in the same strong real form—that is, conjugate by G ((13)). Then there is a* canonical *bijection from equivalence classes of irreducible $(\mathfrak{g}, K_\xi)$-modules to equivalence classes of irreducible $(\mathfrak{g}, K_{\xi'})$-modules.*

Proof. Suppose $g \in G$ conjugates ξ to ξ'. Then twisting by g carries $(\mathfrak{g}, K_\xi)$-modules to $(\mathfrak{g}, K_{\xi'})$-modules. So far this would have worked using just involutive automorphisms θ and θ'. What is special about strong involutions is that *the stabilizer of ξ in G is precisely K_ξ* (whereas the stabilizer of θ_ξ can be bigger). This means that the coset gK_ξ is uniquely determined. Because twisting by K_ξ acts trivially on equivalence classes of $(\mathfrak{g}, K_\xi)$-modules, it follows that the bijection we have defined is unique. □

Using these unique bijections, one can make a well-defined set of equivalence classes of irreducible modules attached to each strong real form $\mathcal{C}$. These equivalence classes are what we will study.

In classical representation theory, one fixes once and for all a Cartan involution θ of G, defining a single symmetric subgroup $K = G^\theta$. The theory of $(\mathfrak{g}, K)$-modules proceeds by defining and studying (for example) various maximal tori preserved by θ. A central idea in the `atlas` algorithms is instead to fix the maximal torus $H \subset G$, and to study various Cartan involutions preserving it. There are hints of this idea in the classical theory. For example, it is common in introductory texts to describe the principal series representations of $SL(2, \mathbb{R})$, because these are closely related to the standard (diagonal) split maximal torus. When discussing the discrete series, it is common to consider instead the (isomorphic) real group $SU(1, 1)$, because the discrete series are closely related to the standard (diagonal) compact maximal torus of $SU(1, 1)$.

In order to pursue this idea, we need to single out the strong involutions preserving our fixed H. These are

$$\widetilde{\mathcal{X}} = \mathcal{I} \cap \mathrm{Norm}_{G\xi_0}(H) = \{\xi \in \mathrm{Norm}_{G\xi_0}(H) \mid \xi^2 \in Z(G)\}$$
$$\mathcal{X} = \widetilde{\mathcal{X}}/H \quad \text{(quotient by conjugation action of } H). \tag{14a}$$

If $z \in Z(G)$, we write

$$\widetilde{\mathcal{X}}_z = \{\xi \in \mathrm{Norm}_{G\xi_0}(H) \mid \xi^2 = z\}$$
$$\mathcal{X}_z = \widetilde{\mathcal{X}}_z/H \quad \text{(quotient by conjugation action of } H) \tag{14b}$$

for the subset of elements of central cocharacter z.

Write $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ for the projection map.

For $x \in \mathcal{X}$ let θ_x be the restriction of θ_ξ to H for any $\xi \in p^{-1}(x)$. The central technical difficulty we face is that *the involution θ_x of H only depends on x, but the extension θ_ξ to G depends on the choice of representative ξ.*

It is easy to check that

$$\theta_x = w_x\xi_0 \in \mathrm{Aut}(H) \quad (w_x \in W(G,H)) \tag{14c}$$

for some *twisted involution* w_x with respect to ξ_0:

$$w_x\xi_0(w_x) = 1. \tag{14d}$$

Conversely, if $w \in W$ is any twisted involution with respect to ξ_0, then

$$\theta_w =_{\text{def}} w\xi_0 \in \mathrm{Aut}(H) \tag{14e}$$

is an involutive automorphism of H (or, equivalently, of $X_*(H)$). We define

$$\mathcal{X}^w = \{x \in \mathcal{X} \mid w_x = w\}, \qquad \widetilde{\mathcal{X}}^w = p^{-1}\mathcal{X}^w, \tag{14f}$$

so that $\mathcal{X}$ is the disjoint union over twisted involutions w of the various $\mathcal{X}^w$.

The definition (14c) of w_x can be restated as

$$\xi = s_1\sigma_{w_x}\xi_0 \qquad (\text{some } s_1 \in H). \tag{14g}$$

Here σ_{w_x} is the Tits group representative of w_x (see (53f)). We call s_1 the *unnormalized torus part* of ξ. We compute

$$\begin{aligned}
\xi^2 &= s_1\sigma_{w_x}\xi_0 s_1\sigma_{w_x}\xi_0 \\
&= s_1\theta_{w_x}(s_1)\sigma_{w_x}\sigma_{\xi_0(w_x)} \\
&= s_1\theta_{w_x}(s_1)\sigma_{w_x}\sigma_{w_x^{-1}} \\
&= s_1\theta_{w_x}(s_1)e((\rho^\vee - \theta_x\rho^\vee)/2) \quad \text{(by Proposition 12.1)} \\
&= (s_1e(-\rho^\vee/2))\theta_{w_x}(s_1e(-\rho^\vee/2))e(\rho^\vee).
\end{aligned} \tag{14h}$$

We call $s = s_1e(-\rho^\vee/2)$ the *normalized torus part* of ξ:

$$\begin{aligned} \xi &= se(\rho^\vee/2)\sigma_{w_x}\xi_0 = s\xi_w \qquad \text{(some } s \in H). \\ \xi^2 &= s\theta_{w_x}(s)z(\rho^\vee). \end{aligned} \tag{14i}$$

Here we have used the definition of ξ_w in the following proposition.

Proposition 3.2. *For every ξ_0-twisted involution $w \in W(G,H)$ there is a basepoint (the one with trivial* normalized *torus part)*

$$\xi_w =_{def} e(\rho^\vee/2)\sigma_w\xi_0 \in \widetilde{\mathcal{X}}$$

of central cocharacter $z(\rho^\vee)$ (see (13c)*):*

$$\xi_w^2 = e(\rho^\vee) = z(\rho^\vee).$$

This basepoint is conjugate by G to the large representative ξ_1 of (11f)*.*

Proof. The formula for ξ_w^2 is immediate from (14h). We omit the argument that ξ_w is conjugate to ξ_1. ☐

Fix a set S of representatives of the set of strong real forms:

$$\widetilde{\mathcal{X}} \supset S \overset{1-1}{\longleftrightarrow} \mathcal{I}/G. \tag{15}$$

Proposition 3.3 ([3, Corollary 9.9]). *There is a canonical bijection*

$$\mathcal{X} \longleftrightarrow \prod_{\xi' \in S} K_{\xi'}\backslash G/B.$$

The bijection restricts to classes on both sides of any fixed central cocharacter (see (13c)*), in which case both sides are finite sets.*

Because of this proposition, we refer to $\mathcal{X}$ as the `KGB`-space, and say $x \in \mathcal{X}$ is a `KGB`-class.

The `KGB` classes are parametrized first by a twisted involution $w \in W$ (see (14c)), and then (for each w) by the allowed (twisted H-conjugacy classes of) normalized torus parts. Our next task is to describe those torus parts. It is convenient to fix also a central element $z \in Z(G)$, and to restrict attention to strong involutions of central cocharacter z. According to (14i), we are therefore seeking to solve the equation

$$s\theta_w(s) = zz(-\rho^\vee) \qquad (\xi = s\xi_w). \tag{16a}$$

Conjugation by $h \in H$ replaces the torus part s by

$$s(h\theta_w(h)^{-1}),$$

so the solutions we want—elements of the `KGB` space $\mathcal{X}$—are cosets of the connected torus

$$(1-\theta_w)H = \text{identity component of } H^{-\theta_w} = H_0^{-\theta_w}. \tag{16b}$$

In order to keep track of such elements, we would like to have nice representatives for the cosets $H/(1-\theta_w)H$. Because the Lie algebra is the direct sum of the $+1$ and -1 eigenspaces of θ_w, we get

$$H = [H_0^{\theta_w}][H_0^{-\theta_w}], \qquad [H_0^{\theta_w}] \cap [H_0^{-\theta_w}] \subset [H_0^{\theta_w}](2) \tag{16c}$$

(the elements of order 2).

This says that every coset of $H_0^{-\theta_w}$ has a representative in $H_0^{\theta_w}$; and that this representative is unique up to multiplication by the finite 2-group $[H_0^{\theta_w}] \cap [H_0^{-\theta_w}]$. We call a coset representative in $H_0^{\theta_w}$ *preferred*. Our immediate goal is therefore to write down all solutions $s \in H_0^{\theta_w}$ of (16a).

As with many calculations in Lie theory, solving this equation is easier on the Lie algebra. We will use the exponential map isomorphisms

$$e\colon \mathfrak{h}/X_*(H) \to H, \qquad e\colon \mathfrak{h}^{\theta_w}/X_*(H)^{\theta_w} \to H_0^{\theta_w} \tag{16d}$$

of (11d). In order to do that, we first choose a logarithm g of the central cocharacter z:

$$z = e(g) \qquad (g \in \mathfrak{h} = X_*(H) \otimes_{\mathbb{Z}} \mathbb{C}). \tag{16e}$$

We say that a strong real form of central cocharacter z has *infinitesimal cocharacter* g. It is convenient (and easy) to arrange also

$$\langle \alpha, g \rangle \in \mathbb{Z}_{>0} \quad (\alpha \in R^+(G,H)). \tag{16f}$$

(Because z is assumed central, roots take integer values on g.)

Next, we choose a logarithm v for the normalized torus part s:

$$s = e(v) \qquad (v \in \mathfrak{h}^{\theta_w}). \tag{16g}$$

Now (16a) can be written

$$2v = v + \theta_w(v) = g - \rho^\vee - \ell \qquad (\text{some } \ell \in X_*(H)), \tag{16h}$$

or

$$v = (g - \rho^\vee - \ell)/2. \tag{16i}$$

Conversely, if $\ell \in X_*(H)$ has the property that

$$g - \rho^\vee + \ell \in \mathfrak{h}^{\theta_w}, \tag{16j}$$

then $e((g - \rho^\vee - \ell)/2)$ is a preferred representative for a normalized torus part (of some $\xi \in \widetilde{\mathcal{X}}$ of central cocharacter z).

We have proven the following proposition.

Proposition 3.4. *Fix an infinitesimal cocharacter g and a ξ_0-twisted involution w. Let $\theta_w = w \circ \xi_0 \in \mathrm{Aut}(H)$. The set $\mathcal{X}_g^w$ of* KGB *classes of infinitesimal cocharacter g (equivalently, of central cocharacter $z = e(g)$) with $w_x = w$ (cf.* (14c)*) is in one-to-one correspondence with*

$$\left\{\bar\ell \in X_*(H)/(1+\theta_w)X_*(H) \mid (1-\theta_w)\ell = (1-\theta_w)(g-\rho^\vee)|\right\}.$$

This set is either empty (if $(1-\theta_w)(g-\rho^\vee)$ does not belong to $(1-\theta_w)X_*$), or has a simply transitive action of

$$X_*^{\theta_w}/(1+\theta_w)X_*.$$

This latter group is a vector space over $\mathbb{Z}/2\mathbb{Z}$, of dimension at most the rank of X_*.

The corresponding x has a preferred representative (cf. (16c)*) ξ with unnormalized torus part*

$$s_1 = e((g-\ell)/2),$$

(see (14g)*) or normalized torus part*

$$s = e((g-\rho^\vee-\ell)/2)$$

(see (14i)*). Here $\ell \in X_*(H)$ is a representative of $\bar\ell$. If we modify the element ℓ in its coset by adding $(1+\theta_w)f$ (for some $f \in X_*(H)$), then s (or s_1) is multiplied by $e((1+\theta_w)f/2)$. That is, this preferred choice of torus part is unique up to the image of $1+\theta_w$ acting on the elements $H(2)$ of order 2 in H. Another formulation is that these preferred representatives ξ of x are a single conjugacy class under $H(2)$.*

The KGB classes in this proposition usually represent several *different* strong real forms (all of a fixed central cocharacter); that is, they are usually *not* conjugate by G. The parametrization of KGB classes is so beautiful and simple precisely because of this inclusion of several real forms. For example, if $G = GL(n)$, $\xi_0 = 1$, and $w = 1$ (so that we are talking about compact maximal tori in equal rank real forms), then the KGB classes amount to discrete series for strong real forms. If we choose $g = \rho^\vee$, then the proposition says that the KGB classes are indexed by $X_*(H)/2X_*(H)$, an n-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. There are $n+1$ different strong real forms appearing in this list: the various $U(p, n-p)$ with $0 \le p \le n$. Such a strong real form has $\binom{n}{p}$ discrete series; only when we take the union over p do we get something as simple as 2^n.

We turn now to writing down Langlands parameters for representations of real forms of G, in the form described in [1]. These are constructed in a manner roughly parallel to the strong involutions above, but in the L-group of (12h) rather than in the extended group for G.

Definition 3.5. A *Langlands parameter* for representations of real forms of G is a pair $({}^\vee\xi, \gamma)$ such that

(a) ${}^\vee\xi \in {}^\vee G {}^\vee\xi_0$;
(b) $\gamma \in {}^\vee\mathfrak{g}$ is semisimple; and
(c) ${}^\vee\xi^2 = e(\gamma)$.

Two Langlands parameters are called *equivalent* if they are conjugate by ${}^\vee G$. The semisimple group element ${}^\vee z = {}^\vee\xi^2 \in {}^\vee G$ is called the *central character* of the Langlands parameter, and the Lie algebra element γ is called the *infinitesimal character*.

Because Langlands parameters matter only up to conjugation by ${}^\vee G$, it is convenient to consider representatives aligned with our fixed ${}^\vee H \subset {}^\vee B$. The Langlands parameter is said to be of *type* ${}^\vee H$ if

$$ {}^\vee\xi \in \mathrm{Norm}_{{}^\vee G^{\vee}\xi_0}({}^\vee H), \quad \text{and} \quad \gamma \in {}^\vee\mathfrak{h}. $$

Finally, a Langlands parameter of type ${}^\vee H$ is said to be *integrally dominant* if it is dominant for the integral root system:

$$ \langle \gamma, {}^\vee\alpha\rangle \in \mathbb{Z} \implies \langle \gamma, {}^\vee\alpha\rangle \geq 0 \quad (\alpha \in R^+(G,H)). \tag{17} $$

Harish-Chandra's theorem guarantees that representation-theoretic infinitesimal characters—homomorphisms from the center of $U(\mathfrak{g})$ to $\mathbb{C}$—are in one-to-one correspondence with ${}^\vee G$ orbits of semisimple elements in ${}^\vee\mathfrak{g}$. The infinitesimal character defined here of the Langlands parameter will turn out to correspond exactly to the representation-theoretic infinitesimal characters of the corresponding representations of real forms of G. Unfortunately the central character defined here bears no such simple relationship to the representation-theoretic central characters.

Here is the original statement of the Langlands classification (with the notion of Langlands parameter modified in accordance with [1]).

Theorem 3.6 ([9, Proposition 4.1]). *In the setting of Definition 3.5, fix a strong real form ξ of G. Attached to each equivalence class of Langlands parameters $({}^\vee\xi, \gamma)$ for G there is a finite set $\Pi_{{}^\vee\xi,\gamma}(\xi)$ of equivalence classes of irreducible $(\mathfrak{g}, K_\xi)$-modules of infinitesimal character γ. These finite sets partition the full set of equivalence classes of such representations.*

Langlands called the finite sets $\Pi_{{}^\vee\xi,\gamma}(\xi)$ *L-packets*, because of their role in automorphic representation theory.

Because of this theorem, we want to understand in more detail what Langlands parameters can look like; and for a fixed Langlands parameter, we want to understand the structure of the L-packet $\Pi_{{}^\vee\xi,\gamma}$.

Proposition 3.7. *Any Langlands parameter is equivalent to an integrally dominant one of type ${}^\vee H$. If the infinitesimal character $\gamma \in {}^\vee\mathfrak{h}$ is regular, then two Langlands parameters of type ${}^\vee H$ and infinitesimal character γ are equivalent (that is, conjugate by ${}^\vee G$) if and only if they are conjugate by ${}^\vee H$. In other words, a collection of all equivalent Langlands parameters of type ${}^\vee H$ and infinitesimal character γ is a single ${}^\vee H$-conjugacy class.*

This is an elementary consequence of the definition, and we omit the proof.

Here is some structure theory for Langlands parameters analogous to that given for strong involutions in (14).

We begin with a dual torus element—not assumed central as in (13c)—

$$ {}^\vee z = e(\gamma) \in {}^\vee H. \tag{18a} $$

We always wish to assume that γ is integrally dominant (17). Almost all of our results will be about the case of *regular* infinitesimal character, so we will assume

$$\langle \gamma, {}^\vee\alpha \rangle \in \mathbb{Z} \implies \langle \gamma, {}^\vee\alpha \rangle > 0 \quad (\alpha \in R^+(G,H)) \tag{18b}$$

or in other words

$$\langle \gamma, \alpha^\vee \rangle \notin \{0, -1, -2, -3, \ldots\} \qquad (\alpha \in R^+(G,H)). \tag{18c}$$

Define

$${}^\vee G({}^\vee z) = \text{centralizer of } {}^\vee z \text{ in } {}^\vee G \supset {}^\vee H. \tag{18d}$$

(This closed reductive subgroup of ${}^\vee G$ may be disconnected, a point which will require some attention; we write ${}^\vee G({}^\vee z)_0$ for its identity component.) An `atlas` *dual strong involution of central character* ${}^\vee z$ is an element

$${}^\vee\xi \in {}^\vee G {}^\vee\xi_0, \qquad {}^\vee\xi^2 = {}^\vee z,$$

and an `atlas` *dual strong real form of central character* ${}^\vee z$ is a ${}^\vee G({}^\vee z)$-conjugacy class ${}^\vee\mathcal{C}$ of such elements. The automorphism

$${}^\vee\theta_{{}^\vee\xi} = \mathrm{int}({}^\vee\xi) \tag{18e}$$

of ${}^\vee G$ preserves ${}^\vee G({}^\vee z)$, and acts on this group (*not* in general on all of ${}^\vee G$) as an involutive automorphism. *It is therefore real forms of* ${}^\vee G({}^\vee z)$ *that are under discussion.*

Keeping in mind the case ${}^\vee z \in {}^\vee H$, we define the *dual* `KGB` *space*—now a space of equivalence classes of Langlands parameters—by

$$\begin{aligned} {}^\vee\widetilde{\mathcal{X}} &= \{{}^\vee\xi \in \mathrm{Norm}_{{}^\vee G {}^\vee\xi_0}({}^\vee H) \mid {}^\vee\xi^2 \in {}^\vee H\} \\ {}^\vee\mathcal{X} &= {}^\vee\widetilde{\mathcal{X}}/{}^\vee H; \end{aligned} \tag{18f}$$

just as in (14a), we are dividing by the conjugation action of ${}^\vee H$. We also write

$$\begin{aligned} {}^\vee\widetilde{\mathcal{X}}_{{}^\vee z} &= {}^\vee\widetilde{\mathcal{X}}_\gamma = \{{}^\vee\xi \in \mathrm{Norm}_{{}^\vee G {}^\vee\xi_0}({}^\vee H) \mid {}^\vee\xi^2 = {}^\vee z = e(\gamma)\}, \\ {}^\vee\mathcal{X}_{{}^\vee z} &= {}^\vee\mathcal{X}_\gamma = {}^\vee\widetilde{\mathcal{X}}_\gamma/{}^\vee H. \end{aligned} \tag{18g}$$

According to Definition 3.5, a Langlands parameter of type ${}^\vee H$ and infinitesimal character γ is a pair $({}^\vee\xi, \gamma)$, with ${}^\vee\xi \in {}^\vee\widetilde{\mathcal{X}}_\gamma$. According to Proposition 3.7, an equivalence class of Langlands parameters of infinitesimal character γ is a pair (y, γ), with $y \in {}^\vee\mathcal{X}_\gamma$.

Associated to $y \in {}^\vee\mathcal{X}$ is an involution of ${}^\vee H$ (conjugation by the element ${}^\vee\xi \in \mathrm{Norm}_{{}^\vee G {}^\vee\xi_0}({}^\vee H)$—that is, the restriction to ${}^\vee H$ of ${}^\vee\theta_{{}^\vee\xi}$—for any representative ${}^\vee\xi$ of y). We denote this involution ${}^\vee\theta_y$:

$${}^\vee\theta_y = {}^\vee w_y {}^\vee\xi_0 \in \mathrm{Aut}({}^\vee H) \simeq \mathrm{Aut}(X^*(H)) \quad ({}^\vee w_y \in W({}^\vee G, {}^\vee H)).$$

Write

$$ {}^\vee\xi = {}^\vee s_1 \sigma_{{}^\vee w_y} {}^\vee\xi_0 \tag{18h} $$

(compare (14g)). The fact that $({}^\vee\theta_y)^2 = 1$ is equivalent to

$$ {}^\vee w_y {}^\vee\xi_0({}^\vee w_y) = 1, \tag{18i} $$

i.e., ${}^\vee w_y$ is a twisted involution (in W) with respect to the automorphism ${}^\vee\xi_0$. Just as in (14f), we define

$$ {}^\vee\mathcal{X}^{{}^\vee w} = \{y \in {}^\vee\mathcal{X} \mid {}^\vee w_y = {}^\vee w\}. \tag{18j} $$

Exactly as in Proposition 3.4, we can now describe the set of Langlands parameters attached to a given twisted involution.

Proposition 3.8. *Fix an infinitesimal character γ and a ${}^\vee\xi_0$-twisted involution ${}^\vee w$. Let ${}^\vee\theta_{{}^\vee w} = {}^\vee w \circ \xi_0 \in \mathrm{Aut}(H)$. The set $\mathcal{X}_\gamma^{{}^\vee w}$ of dual* KGB *classes of infinitesimal character γ (equivalently, of central character ${}^\vee z = e(\gamma)$) with ${}^\vee w_y = {}^\vee w$ (cf.* (18h)*) is in one-to-one correspondence with*

$$ \left\{\overline{\lambda} \in X^*(H)/(1 + {}^\vee\theta_{{}^\vee w})X^*(H) \mid (1 - {}^\vee\theta_{{}^\vee w})\lambda = (1 - {}^\vee\theta_{{}^\vee w})(\gamma - \rho)|\right\}. $$

This set is either empty (if $(1 - {}^\vee\theta_{{}^\vee w})(\gamma - \rho)$ does not belong to $(1 - {}^\vee\theta_{{}^\vee w})X^*$), or has a simply transitive action of

$$ (X^*)^{{}^\vee\theta_{{}^\vee w}}/(1 + {}^\vee\theta_{{}^\vee w})X^*. $$

This latter group is a vector space over $\mathbb{Z}/2\mathbb{Z}$, of dimension at most the rank of X^*.

The corresponding y has a preferred representative (defined by analogy with (16c)*) ${}^\vee\xi$ with unnormalized torus part*

$$ {}^\vee s_1 = e((\gamma - \lambda)/2), $$

(see (18h)*) or normalized torus part*

$$ {}^\vee s = e((\gamma - \rho - \lambda)/2) $$

(defined by analogy with (14i)*). Here $\lambda \in X^*(H)$ is a representative of $\overline{\lambda}$. If we modify the element λ in its coset by adding $(1 + {}^\vee\theta_{{}^\vee w})\phi$ (for some $\phi \in X^*(H)$), then ${}^\vee s$ (or ${}^\vee s_1$) is multiplied by $e((1 + {}^\vee\theta_{{}^\vee w})\phi/2)$. That is, this preferred choice of torus part is unique up to the image of $1 + {}^\vee\theta_{{}^\vee w}$ acting on the elements ${}^\vee H(2)$ of order 2 in ${}^\vee H$. Another formulation is that these preferred representatives ${}^\vee\xi$ of y are a single conjugacy class under ${}^\vee H(2)$.*

The proof is identical to that of Proposition 3.4, and we omit it.

Because the set is parametrized by certain (cosets of) characters of H, it is easy and useful to reformulate the result as follows.

Corollary 3.9. *In the setting of Proposition 3.8, the set of dual* KGB *classes* ${}^\vee\mathcal{X}_\gamma^{{}^\vee w}$ *is naturally in bijection with the set of (automatically one-dimensional) irreducible* $(\mathfrak{h}, H^{\theta_w})$*-modules of differential equal to* $\gamma - \rho$*.*

Proof. By definition, an $(\mathfrak{h}, H^{\theta_w})$-module is a vector space carrying an algebraic action of the group H^{θ_w}, and a representation of the abelian Lie algebra $\mathfrak{h}$, so that the differential of the former is the restriction to $\mathfrak{h}^{\theta_w}$ of the latter. In the corollary, we want $\mathfrak{h}$ to act by $\gamma - \rho$, so there is nothing to say about that. The characters of H^{θ_w} are the restrictions to H^{θ_w} of characters of H; so they are indexed by

$$\overline{\lambda} \in X^*(H)/(1 - {}^t\theta_w)X^*(H), \tag{19}$$

the denominator being the characters trivial on H^{θ_w}. Now it is clear that the modules we want are indexed exactly by such cosets $\overline{\lambda}$, subject to the requirement

$$(1 + {}^t\theta_w)\lambda = (1 + {}^t\theta_w)(\gamma - \rho)$$

(that the differential of λ is the restriction of $\gamma - \rho$). This is exactly the condition on λ written in Proposition 3.8. □

Definition 3.10. A KGB class x and a dual KGB class y are said to be *aligned* if

$$-{}^t\theta_x = {}^\vee\theta_y \in \mathrm{Aut}(X^*(H))$$

((14c), (18h)). Equivalently, we require the twisted involutions to satisfy

$$w_x w_0 = {}^\vee w_y.$$

In this case we call the pair (x, y) an atlas *parameter* for G. We write

$$\mathcal{Z} = \{(x, y) \in \mathcal{X} \times {}^\vee\mathcal{X} \mid -{}^t\theta_x = {}^\vee\theta_y\},$$

for the set of all atlas parameters. If $z = e(g) \in Z(G)$ and ${}^\vee z = e(\gamma) \in {}^\vee H$ (with g and γ regular and integrally dominant), we write

$$\mathcal{Z}_{z,{}^\vee z} = \mathcal{Z}_{g,\gamma} = \{(x, y) \in \mathcal{Z} \mid x^2 = z,\ y^2 = {}^\vee z\}$$

for the subset of parameters of infinitesimal cocharacter g and infinitesimal character γ. If $w \in W$ is a ξ_0-twisted involution, we define ${}^\vee w = ww_0$ (a ${}^\vee\xi_0$-twisted involution, and write

$$\mathcal{Z}^w = \mathcal{X}^w \times {}^\vee\mathcal{X}^{{}^\vee w},$$

so that $\mathcal{Z}$ is the disjoint union over ξ_0-twisted involutions w of the subsets $\mathcal{Z}^w$.

We are now in a position to sharpen the Langlands classification Theorem 3.6, by parametrizing each L-packet.

Theorem 3.11 ([1, Theorem 1.18]). *In the setting of Definition 3.5, fix a regular and integrally dominant infinitesimal character* $\gamma \in {}^\vee\mathfrak{h}$*, and a regular integral dominant*

$g \in \mathfrak{h}$ (so that $e(g) \in Z(G)$). Then there is a natural bijection between irreducible admissible representations of infinitesimal character γ of strong real forms of G having infinitesimal cocharacter g; and the set of pairs $(x, y) \in \mathcal{Z}_{g,\gamma}$ of `atlas` *parameters of infinitesimal cocharacter g and infinitesimal character γ. In this bijection, the strong real form may be taken to be any representative ξ of the first factor x. The Langlands parameter (Definition 3.5 may be taken to be $({}^\vee\xi, \gamma)$, with ${}^\vee\xi$ any representative of the second factor y. We write*

$$J(x, y, \gamma)$$

for the irreducible module of infinitesimal character γ attached to (x, y).

The notation requires some explanation, because we have not even said of what group $J(x, y, \gamma)$ is a representation. If ξ is any representative of x, then $\theta_\xi = \mathrm{int}(\xi)$ is a well-defined involutive automorphism of G, with fixed point group K_ξ as in (13b). Then $J(x, y, \gamma)$ is an irreducible $(\mathfrak{g}, K_\xi)$-module. A different choice ξ' of representative of x gives rise to a (necessarily different, because $K_{\xi'}$ is different) $(\mathfrak{g}, K_{\xi'})$-module $J'(x, y, \gamma)$. Part of what the theorem means is that these two different modules are identified by the canonical bijection of Proposition 3.1.

Corollary 3.12. *Suppose $({}^\vee\xi, \gamma)$ is a Langlands parameter of type ${}^\vee H$ and regular infinitesimal character γ. Fix a dominant regular infinitesimal cocharacter g as in* (16f). *Then the union (over strong real forms of infinitesimal cocharacter g) of the L-packets $\Pi_{{}^\vee\xi,\gamma}(\xi)$ (Theorem 3.6) may be identified with the set $\mathcal{X}_g^w$ of Proposition 3.4. (Here w is the twisted involution dual to the one for ${}^\vee\xi$.) In particular, this L-packet is either empty (if $(1-\theta_w)(g-\rho^\vee)$ does not belong to $(1-\theta_w)X_*$), or has a simply transitive action of*

$$X_*^{\theta_w}/(1+\theta_w)X_*.$$

Notice that if we consider real forms of infinitesimal cocharacter $\rho^\vee$ (which includes the quasisplit real form), then this union of L-packets is never empty. This is consistent with Langlands' result that every L-packet is nonempty for the quasisplit real form.

This classical corollary is the most familiar way of thinking about ambiguity in the Langlands classification: starting with a Langlands parameter, and enumerating the various (strong) real forms where it can give a representation. We are in fact going to be interested mostly in the *dual* problem: starting with a strong real form of type H and and an infinitesimal character γ, and enumerating the various Langlands parameters giving a representation. For example, if we start with a split maximal torus, then the Langlands parameters in question just index the characters of the split maximal torus of differential γ. These admit a simply transitive action of the group $(\mathbb{Z}/2\mathbb{Z})^n$ of characters of the component group of the split torus.

Corollary 3.13. *Suppose ξ is a strong real form of type H and dominant regular infinitesimal cocharacter g. Fix a dominant regular infinitesimal character $\gamma \in \mathfrak{h}^*$.*

Then the collection of Langlands parameters $({}^\vee\xi,\gamma)$ of type ${}^\vee H$ aligned with ξ (Definition 3.10) may be identified with the set ${}^\vee\mathcal{X}_\gamma^{{}^\vee w}$ of Proposition 3.4. (Here ${}^\vee w = ww_0$ is the twisted involution dual to $w = w_\xi$.) In particular, this set of parameters is either empty (if $(1-{}^\vee\theta_{{}^\vee w})(\gamma-\rho)$ does not belong to $(1-{}^\vee\theta_{{}^\vee w})X^*$), or has a simply transitive action of

$$(X^*)^{{}^\vee\theta_{{}^\vee w}}/(1+{}^\vee\theta_{{}^\vee w})X^*.$$

We are going to need a slightly more precise understanding of how the parametrization of representations in Theorem 3.11 actually works. So let us fix an `atlas` parameter (x,y) (Definition 3.10) of (integrally dominant regular) infinitesimal character $\gamma\in\mathfrak{h}^*$. Choose a strong real form representative ξ for x, so that what we are seeking to construct is an irreducible $(\mathfrak{g},K_\xi)$-module $J(x,y,\gamma)$. The construction begins with the θ_ξ-stable Cartan subgroup H. The Cartan involution θ_ξ acts on H by θ_w, so

$$H\cap K_\xi = H^{\theta_w}. \tag{20a}$$

By definition of `atlas` parameter, $y\in{}^\vee\mathcal{X}_\gamma^{{}^\vee w}$; so by Corollary 3.9, y defines

$$\mathbb{C}(y,\gamma) = \text{irreducible } (\mathfrak{h},H\cap K_\xi)\text{-module of differential } \gamma-\rho. \tag{20b}$$

We want to construct a $(\mathfrak{g},K_\xi)$-module using the character $\mathbb{C}(y,\gamma)$. This is a large and complicated problem, solved by work of Zuckerman reported in [14], but here is a sketch. (Shorthand for this construction is *cohomological induction*, and we will use that phrase to refer to it.)

Now we extend $\mathbb{C}(y,\gamma)$ to a $(\mathfrak{b},H\cap K_\xi)$-module by making $\mathfrak{n}$ act trivially, and then form the (dual to Verma) $(\mathfrak{g},H\cap K_\xi)$-module

$$M(y,\gamma) = \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}),\mathbb{C}(y,\gamma)\otimes\mathbb{C}(2\rho)). \tag{20c}$$

Here $\mathbb{C}(2\rho)$ is the representation of B on the top exterior power of the Lie algebra: the sum of the positive roots. The weight of $\mathfrak{h}$ by which we are “producing” is $\gamma+\rho$, so this is the lowest weight of $M(y,\gamma)$. By the theory of Verma modules, $M(y,\gamma)$ has infinitesimal character γ. Now we apply the Zuckerman right derived functor ([8, (2.113)])

$$\left(\Gamma^{\mathfrak{g},K_\xi}_{\mathfrak{g},H\cap K_\xi}\right)^S : (\mathfrak{g},H\cap K_\xi)\text{-modules} \to (\mathfrak{g},K_\xi)\text{-modules}, \tag{20d}$$

with $S=\dim\mathfrak{n}\cap\mathfrak{k}_\xi$, obtaining what is called the *standard* $(\mathfrak{g},K_\xi)$*-module*

$$I^{\text{quo}}(x,y,\gamma) = \left(\Gamma^{\mathfrak{g},K_\xi}_{\mathfrak{g},H\cap K_\xi}\right)^S (M(y,\gamma)). \tag{20e}$$

This module has finite length, and has a unique irreducible quotient $J(x,y,\gamma)$. Proofs may be found in [8, Theorem 11.129].

4 Twisting parameters

We want to consider the action of δ_0 on representations. In terms of parameters, we need to study the action of δ_0 on $\mathcal{X}$ (and ${}^\vee\delta_0$ on ${}^\vee\mathcal{X}$). We do this in the setting of Propositions 3.8 and 3.4.

Of course a δ_0-fixed representation of infinitesimal character γ can exist only if the infinitesimal character γ is itself fixed by δ_0; that is, if and only if

$$ {}^\vee\delta_0(\gamma) = z \cdot \gamma \qquad \text{(some } z \in W\text{).} \tag{21} $$

Because of the integrally dominant condition (18b) that we impose on γ (and which is automatically inherited by ${}^\vee\delta_0(\gamma)$), it follows that

$$ z \cdot R^+(\gamma) = R^+({}^\vee\delta_0(\gamma)) \subset R^+; \tag{22} $$

here $R^+(\gamma)$ is the set of positive integral roots defined by γ. If γ is integral (so that $R^+(\gamma) = R^+$), such a condition forces $z = 1$, i.e., ${}^\vee\delta_0(\gamma) = \gamma$. If γ is *not* integral, however, we can draw no such conclusion. Here is a convenient substitute.

Lemma 4.1. *Suppose $\gamma \in \mathfrak{h}^*$, and that ${}^\vee\delta_0$ preserves the W orbit of γ. Then this orbit has an integrally dominant representative γ' with the property that*

$$ {}^\vee\delta_0(\gamma') = \gamma'. $$

We omit the (elementary) proof. Because of this lemma, it is sufficient to study representations of infinitesimal character represented by a ${}^\vee\delta_0$-fixed integrally dominant weight

$$ {}^\vee\delta_0(\gamma) = \gamma, \quad \langle \gamma, \beta^\vee \rangle \notin \{0, -1, -2, \dots\} \quad (\beta \in R^+(G,H)). \tag{23} $$

The situation for real forms is a bit more subtle, because the infinitesimal cocharacter is not an invariant of a real form, but merely a useful extra parameter that we attach to the real form. Here is what we would like to know.

Desideratum 4.2. Suppose ξ is a strong involution for G (see (13a)), of (dominant regular integral) infinitesimal cocharacter g. Assume that the involution $\delta_0 \circ \theta_\xi \circ \delta_0$ is equivalent (that is, conjugate by G) to θ_ξ. Then there is a δ_0-fixed regular integral g', and a strong involution ξ' of infinitesimal cocharacter g', such that $\theta_\xi = \theta_{\xi'}$.

Unfortunately this statement is false.

Example 4.3. Suppose $G = SL(4)$, endowed with the trivial distinguished automorphism ξ_0 (so that we are considering equal rank real forms) and the nontrivial distinguished automorphism δ_0. On the diagonal torus,

$$ \delta_0(a_1, a_2, a_3, a_4) = (a_4^{-1}, a_3^{-1}, a_2^{-1}, a_1^{-1}). $$

Let $\omega = \exp(2\pi i/8)$ be a primitive eighth root of 1, and define

$$\xi = \mathrm{diag}(\omega, \omega, \omega, \omega^5) = \omega[\mathrm{diag}(1, 1, 1, -1)].$$

Then $\xi^2 = \omega^2 I = iI$, a central element of order 4; and

$$K_\xi = S(GL(3) \times GL(1)),$$

the complexified maximal compact subgroup for the real form $SU(3,1)$ of G. The infinitesimal cocharacter of this strong real form is any weight of the form

$$g = (g_1, g_2, g_3, g_4) \in \mathbb{Q}^4,$$
$$\sum g_j = 0, \quad g_1 > g_2 > g_3 > g_4,$$
$$\exp(2\pi i g_j) = \xi_j;$$

that is,

$$g_j \cong \begin{cases} 1/8 \pmod{\mathbb{Z}} & (g = 1, 2, 3) \\ 5/8 \pmod{\mathbb{Z}} & (g = 4), \end{cases}$$

These conditions are easily satisfied (for example by $(17/8, 9/8, 1/8, -27/8)$); but it is easy to see that they *cannot* be satisfied by a δ_0-fixed g. The reason is that

$$\delta_0(g) = (-g_4, -g_3, -g_2, -g_1).$$

If this is equal to g, then $g_3 = -g_2$, which contradicts the requirements $g_3 \cong g_2 \cong 1/8 \pmod{\mathbb{Z}}$. On the other hand, $\delta_0 \circ \theta_\xi \circ \delta_0$ *is* conjugate to θ_ξ (by a cyclic permutation matrix).

One might hope that in this example none of the representations of $SU(3,1)$ is fixed by δ_0, and indeed none of the four discrete series representations is fixed; but there is a spherical principal series representation (of infinitesimal character ρ) which *is* fixed.

In any case, we are going to consider only cases when Desideratum 4.2 is true; that is, we are going to consider only real forms of G of infinitesimal cocharacter g satisfying

$$\delta_0(g) = g, \quad \langle \beta, g \rangle \notin \{0, -1, -2, \dots\} \quad (\beta \in R^+(G, H)). \tag{24}$$

In general there will be an extra twist by the central element z, ($-I$ in the example), satisfying $\delta_0(\xi^2) = z\xi^2$.

Suppose $x \in \mathcal{X}$, and $\xi \in p^{-1}(x)$. Then $\delta_0(x) = x$ if and only if $\delta_0(\xi) = h^{-1}\xi h$ for some $h \in H$ (we cannot necessarily choose ξ so that $h = 1$). This is equivalent to

$$\xi(h\delta_0)\xi^{-1} = h\delta_0,$$

i.e.,

$$({}^{\delta_0}H)^{\theta_\xi} = \langle H^{\theta_x}, h\delta_0 \rangle.$$

Suppose ξ corresponds to $\overline{\ell} \in X_*(H)/(1+\theta_x)X_*(H)$ by Proposition 3.4, and $\ell \in X_*(H)$ is a representative. Then x is δ_0-fixed if and only if

$$\delta_0\ell \in \ell + (1+\theta_x)X_*(H).$$

Here is a precise statement.

Proposition 4.4. *Suppose g is an infinitesimal cocharacter as in* (16f). *Suppose $x \in \mathcal{X}$ has infinitesimal cocharacter g, and let $w = w_x$ be the underlying twisted involution* (14c). *Assume*

$$\delta_0(g) = g, \quad \delta_0(w) = w. \tag{24a}$$

Suppose that x corresponds via Proposition 3.4 to $\overline{\ell} \in X_*(H)/(1+\theta_x)X_*(H)$. Choose $\ell \in X_*(H)$ representing $\overline{\ell}$, so x has a representative with unnormalized torus part $s_1 = e((g-\ell)/2)$, or normalized torus part $s = e((g-\ell-\rho^\vee)/2)$.

1. *The class x is fixed by δ_0 if and only if*

$$(\delta_0 - 1)\ell = (1+\theta_x)t \qquad \textit{(some } t \in X_*(H)\textit{)}. \tag{24b}$$

2. *The element t is uniquely defined by ℓ up to adding $X_*(H)^{-\theta_x}$; if also ℓ is modified in its coset, then t changes by $(1-\delta_0)X_*(H)$.*
3. *The corresponding special representative*

$$\xi = e((g-\ell)/2)\sigma_w\xi_0 \tag{24c}$$

satisfies

$$\delta_0\xi\delta_0^{-1} = e((1-\theta_x)t/2)\xi = e(t/2)\xi e(-t/2); \tag{24d}$$

that is, ξ is conjugate to its δ_0 twist using the element $e(t/2)$ of $H(2)$.

4. *Condition* (24d) *is equivalent to*

$$({}^{\delta_0}H)^{\theta_\xi} = \langle H^{\theta_x}, e(-t/2)\delta_0\rangle. \tag{24e}$$

Here is the version for ${}^\vee G$.

Proposition 4.5. *Suppose γ is an infinitesimal character as in* (18b). *Suppose $y \in {}^\vee\mathcal{X}$ has infinitesimal character γ, and let $w = w_y$ be the underlying twisted involution* (18h). *Assume*

$${}^\vee\delta_0(\gamma) = \gamma, \quad {}^\vee\delta_0(w) = w. \tag{25a}$$

Suppose that y corresponds via Proposition 3.8 to $\overline{\lambda} \in X^*(H)/(1+\theta_x)X^*(H)$. Choose $\lambda \in X^*(H)$ representing $\overline{\lambda}$, so y has a representative with unnormalized torus part $e((\gamma-\lambda)/2)$, and normalized torus part $e((\gamma-\lambda-\rho)/2)$.

1. *The class y is fixed by ${}^\vee\delta_0$ if and only if*

$$({}^\vee\delta_0 - 1)(\lambda) = (1+{}^\vee\theta_y)\tau \qquad \textit{(some } \tau \in X^*(H)\textit{)}. \tag{25b}$$

2. *The element τ is uniquely defined by λ up to adding $X^*(H)^{-{}^\vee\theta_y}$; if also ℓ is modified in its coset, then* t *changes by* $(1-\delta_0)X^*(H)$.
3. *The corresponding special representative*

$$^\vee\xi = e((\gamma-\lambda)/2)^\vee\sigma_w{}^\vee\xi_0 \tag{25c}$$

satisfies

$$^\vee\delta_0(^\vee\xi)^\vee\delta_0^{-1} = e((1-{}^\vee\theta_y)\tau/2) = e(\tau/2)^\vee\xi e(-\tau/2)\xi; \tag{25d}$$

that is, $^\vee\xi$ is conjugate to its $^\vee\delta_0$ twist using the element $e(\tau/2)$ *of* $^\vee H(2)$.
4. *Condition* (24d) *is equivalent to*

$$\left(^{^\vee\delta_0\,\vee}H\right)^{^\vee\theta_{^\vee\xi}} = \langle H^{^\vee\theta_y}, e(-\tau/2)^\vee\delta_0\rangle. \tag{25e}$$

5 Extended parameters

We now define parameters for $(\mathfrak{g}, {}^{\delta_0}K)$-modules. Suppose $(x,\overline{\lambda},\gamma)$ is a δ_0-fixed parameter. If $\xi \in p^{-1}(x) \in \widetilde{\mathcal{X}}$, then $J(x,\overline{\lambda},\gamma)$ is a δ_0-fixed $(\mathfrak{g},K_\xi)$-module. As discussed in the introduction this can be extended in two ways to give a $(\mathfrak{g}, {}^{\delta_0}K_\xi)$-module.

Lemma 5.1. *Suppose $(x,\overline{\lambda},\gamma)$ is a δ_0-fixed parameter. Choose $h\delta_0 \in ({}^{\delta_0}H)^{\theta_\xi}$ as in Proposition 4.4. The two extensions of $J(x,\overline{\lambda},\nu)$ to a $(\mathfrak{g}, {}^{\delta_0}K_\xi)$-module are parametrized by the two extensions of the character $\overline{\lambda}$ of H^{θ_x} to*

$$({}^{\delta_0}H)^{\theta_\xi} = \langle H^{\theta_x}, h\delta_0\rangle,$$

whose values at $h\delta_0$ are the two square roots of $\overline{\lambda}(h\delta_0(h))$.

We now begin to assemble the data—the *extended parameters* of Definition 5.4—that we will use to construct one of the square roots required in Lemma 5.1. We will consider representations with a fixed regular infinitesimal character, for real forms with a fixed infinitesimal cocharacter. So fix an integrally dominant infinitesimal character γ:

$$\gamma \in X^*(H)_{\mathbb{C}} \subset \mathfrak{h}^*, \quad \langle\gamma, {}^\vee\alpha\rangle \notin \mathbb{Z}_{<0} \quad (\alpha \in R^+(G,H)) \tag{26a}$$

and an integral dominant infinitesimal cocharacter g:

$$g \in X_*(H)_{\mathbb{Q}} \subset \mathfrak{h}, \quad \langle g,\alpha\rangle \in \mathbb{Z}_{>0} \quad (\alpha \in R^+(G,H)). \tag{26b}$$

We require (see Lemma 4.1 and Conjecture 4.2)

$$\delta_0(g) = g, \quad {}^t\delta_0(\gamma) = \gamma. \tag{27}$$

Definition 5.2. Suppose (x, y, γ) is a parameter for a δ_0-fixed representation. Define $\overline{\lambda} \in X^*(H)/(1-\theta_x)X^*(H) \simeq X^*(H^{\theta_x})$ (from y) by Proposition 3.8. Choose a preferred representative ξ for x, and define $\overline{\ell} \in X_*(H)/(1+\theta_x)X_*(H)$ corresponding to ξ, by Proposition 3.4. Choose a representative $\ell \in X_*(H)$ for $\overline{\ell}$, and choose $t \in X_*(H)$ satisfying (24b). Set $h = e(t/2)$ so $h\delta_0 \in ({}^{\delta_0}H)^{\theta_\xi}$ and $(h\delta_0)^2 = h\delta_0(h) \in H^{\theta_x}$. Define

$$\epsilon(x, y) = \overline{\lambda}(h\delta_0(h)). \tag{27a}$$

Lemma 5.3.

1. $\epsilon(x, y) = (-1)^{\langle\overline{\lambda},(1+\delta_0)t\rangle}$,
2. $\epsilon(x, y)$ *is independent of the choices of* ξ, $\overline{\ell}$, ℓ, *and* t *(for fixed* g *and* γ*).*

Proof. The first statment is immediate. By (24b), t is determined by ℓ up to adding elements of $X_*(H)^{-\theta_x}$, and by $\overline{\ell}$ up to $(1-\delta_0)X_*(H)$. Therefore

$$t \text{ is determined by } x \text{ up to adding } X_*(H)^{-\theta_x} + (1-\delta_0)X_*(H).$$

We have

$$\begin{aligned}(-1)^{\langle\lambda,(1+\delta_0)t\rangle} &= (-1)^{\langle\lambda,(1\pm\delta_0)t\rangle} \\ &= (-1)^{\langle(1\pm{}^t\delta_0)\lambda,t\rangle} \\ &= (-1)^{\langle(1\pm{}^\vee\theta_y)\tau,t\rangle} \\ &= (-1)^{\langle\tau,(1\pm\theta_x)t\rangle}.\end{aligned} \tag{28}$$

The second equality shows this sign is unchanged by adding to t an element of $(1-\delta_0)X_*(H)$, and the last one shows it is unaffected by adding elements of $X_*(H)^{-\theta_x}$. ☐

We need to choose a square root of $\epsilon(x, y)$. Just as for the parameters (x, y) for representations of real forms of G, it is helpful to symmetrize the picture with respect to G and ${}^\vee G$.

Definition 5.4. Fix $\gamma,\ g$ as in (26), and a ξ_0-twisted involution $w \in W$. Let $\theta = \theta_w = w\xi_0 \in \mathrm{Aut}(H)$ and ${}^\vee\theta = {}^\vee\theta_{ww_0} = -{}^t\theta$.

An *extended parameter* (for the twisted involution w and the specified infinitesimal character and cocharacter) is a set

$$E = (\lambda, \tau, \ell, t)$$

where

1. $\lambda \in X^*(H)$ satisfies $(1-{}^\vee\theta)\lambda = (1-{}^\vee\theta)(\gamma-\rho)$;
2. $\ell \in X_*(H)$ satisfies $(1-\theta)\ell = (1-\theta)(g-{}^\vee\rho)$;
3. $\tau \in X^*(H)$ satisfies $({}^\vee\delta_0-1)\lambda = (1+{}^\vee\theta)\tau$;
4. $t \in X_*(H)$ satisfies $(\delta_0-1)\ell = (1+\theta)t$.

Associated to an extended parameter $E = (\lambda, \tau, \ell, t)$ are the following elements:

(a) $\xi(E) \in \widetilde{\mathcal{X}}$ corresponds to λ by Proposition 3.8;
(b) ${}^\vee\xi(E) \in \widetilde{{}^\vee\mathcal{X}}$ corresponds to ℓ by Proposition 3.4;
(c) $x(E) =_{\text{def}} p(\xi(E)) \in \mathcal{X}$, $x(E)^2 = e(g)$;
(d) $y(E) =_{\text{def}} p({}^\vee\xi(E)) \in {}^\vee\mathcal{X}$, $y(E)^2 = e(\gamma)$;
(e) $h(E)\delta_0 = e(t/2)\delta_0 \in ({}^{\delta_0}H)^{\theta_\xi}$ (cf. (24e));
(f) ${}^\vee h(E){}^\vee\delta_0 = e(\tau/2){}^\vee\delta_0 \in \left({}^{\vee\delta_0\vee}H\right)^{\vee\theta_{\vee\xi}}$ (cf. (25e)).

We say E is an *extended parameter for* $(x(E), y(E))$.

Definition 5.5. Suppose (λ, τ, ℓ, t) is an extended parameter for (x, y). Define

$$z(\lambda, \tau, \ell, t) = i^{\langle \tau, (1+\theta_x)t\rangle}(-1)^{\langle \lambda, t\rangle}. \tag{29}$$

By (28) we have

$$z(\lambda, \tau, \ell, t)^2 = \epsilon(x, y). \tag{30}$$

Associated to (λ, τ, ℓ, t) is an extension of $J(x, y, \gamma)$ defined as follows.

Definition 5.6. Suppose (λ, τ, ℓ, t) is an extended parameter for (x, y). Set $\xi = \xi(\lambda, \tau, \ell, t)$ and $h = h(\lambda, \tau, \ell, t) = e(t/2)$. Define an extension of $\overline{\lambda}$ to $({}^{\delta_0}H)^{\theta_\xi}$ (see Lemma 5.1) by having it take the value $z(\lambda, \tau, \ell, t)$ at $h\delta_0$. This defines an extension of $J(x, y, \gamma)$ to a $(\mathfrak{g}, {}^{\delta_0}K_\xi)$-module, denoted $J_z(\lambda, \tau, \ell, t)$. (The subscript z refers to the particular formula chosen in Definition 5.5.)

We deal with the question of equivalence of parameters in the next section.

For later use we record precisely how these elements depend on the various choices. Suppose we are given $(x, y) \in \mathcal{Z}$. Choose representatives ξ for x and ${}^\vee\xi$ for y by Propositions 3.4 and 3.8, respectively. That is

$$\begin{aligned} \xi &= e((g-\ell)/2)\sigma_w\xi_0 \\ {}^\vee\xi &= e((\gamma-\lambda)/2){}^\vee\sigma_{ww_0}{}^\vee\xi_0. \end{aligned} \tag{31}$$

Then

$$\begin{aligned} &\ell \text{ is determined by } \xi \text{ up to } 2X_*^{\theta_x} \\ &\ell \text{ is determined by } x \text{ up to } (1+\theta_x)X_* \\ &\lambda \text{ is determined by } {}^\vee\xi \text{ up to } 2(X^*)^{\vee\theta_y} \\ &\lambda \text{ is determined by } y \text{ up to } (1+{}^\vee\theta_y)X^*. \end{aligned} \tag{32}$$

It is helpful to write in addition

$$\begin{aligned} f &= (\delta_0 - 1)\ell = (1+\theta_x)t \\ \phi &= ({}^\vee\delta_0 - 1)\lambda = (1+{}^\vee\theta_y)\tau. \end{aligned} \tag{33}$$

Because (for example) t is evidently determined by f up to $X_*^{-\theta_x}$, the corresponding uniqueness statements are

$$\begin{aligned}
&f \text{ is determined by } \xi \text{ up to } 2(1-\delta_0)X_*^{\theta_x}\\
&f \text{ is determined by } x \text{ up to } (1-\delta_0)(1+\theta_x)X_*\\
&t \text{ is determined by } \xi \text{ up to } (1-\delta_0)X_*^{\theta_x} + X_*^{-\theta_x}\\
&t \text{ is determined by } x \text{ up to } (1-\delta_0)(1+\theta_x)X_* + X_*^{-\theta_x}\\
&\phi \text{ is determined by } {}^\vee\xi \text{ up to } 2(1-{}^\vee\delta_0)(X^*)^{{}^\vee\theta_y}\\
&\phi \text{ is determined by } y \text{ up to } (1-{}^\vee\delta_0)(1+{}^\vee\theta_y)X^*\\
&\tau \text{ is determined by } {}^\vee\xi \text{ up to } (1-{}^\vee\delta_0)(X^*)^{{}^\vee\theta_y} + (X^*)^{-{}^\vee\theta_y}\\
&\tau \text{ is determined by } y \text{ up to } (1-{}^\vee\delta_0)(1+{}^\vee\theta_y)X^* + (X^*)^{-{}^\vee\theta_y}.
\end{aligned} \tag{34}$$

Eventually we will want a parallel choice of square root of ϵ related to the dual group ${}^\vee G$. This is

$$\begin{aligned}
\zeta(\lambda,\tau,\ell,t) &=_{\text{def}} i^{\langle\tau,f\rangle}(-1)^{\langle\tau,\ell\rangle}\\
&= z(\lambda,\tau,\ell,t)(-1)^{\langle\lambda,t\rangle}(-1)^{\langle\tau,\ell\rangle}.
\end{aligned} \tag{35}$$

6 Equivalences of extended parameters

In this section we record how to tell when two of the extended modules defined in Definition 5.6 are equivalent.

Fix γ and g as usual, and suppose (x,y) is a δ_0-fixed parameter. Choose two extended parameters

$$E = (\lambda,\tau,\ell,t), \qquad E' = (\lambda',\tau',\ell',t') \tag{36a}$$

for (x,y) (Definition 5.4). Set $\xi = \xi(E)$, $\xi' = \xi(E')$, and define

$$K_\xi = \mathrm{Cent}_G(\xi), \qquad {}^{\delta_0}K_\xi = \mathrm{Cent}_{\langle G,\delta_0\rangle}(\xi), \tag{36b}$$

and similarly with primes. Because ξ and ξ' are assumed to be conjugate by G, Proposition 3.1 provides a *canonical* identification

$$\text{irreducible } (\mathfrak{g},K_\xi)\text{-modules} \simeq \text{irreducible } (\mathfrak{g},K_{\xi'})\text{-modules} \tag{36c}$$

(by twisting the action by $\mathrm{Ad}(g)$). Exactly the same argument applies to irreducible $(\mathfrak{g},{}^{\delta_0}K_\xi)$-modules.

Definition 6.1. We say E *is equivalent to* E' if $J_z(E)$ and $J_z(E')$ correspond by this canonical identification.

Define $\mathrm{sgn}(E,E') = 1$ if $J_z(E) \sim J_z(E')$, or -1 otherwise.

In other words, if [ ] denotes the image of a representation of an extended group in the module $\mathcal{M}$ (see the Introduction or Section 7), then

$$[J_z(E)] = \mathrm{sgn}(E, E')[J_z(E')]. \tag{36d}$$

The Langlands classification attaches to (ξ, y) an irreducible $(\mathfrak{g}, K_\xi)$-module $J(\xi, y)$. The construction of $J(\xi, y, \gamma)$ begins with a one-dimensional $(\mathfrak{h}, H^{\theta_\xi})$-module $\mathbb{C}_{y,\gamma}$. Cohomological induction produces a "standard" $(\mathfrak{g}, K_\xi)$-module $I(\xi, y, \gamma)$, with unique irreducible quotient $J(\xi, y, \gamma)$. The nature of this construction makes it obvious that the identification of (36c) carries $I(\xi, y, \gamma)$ to $I(\xi', y, \gamma)$, and consequently $J(\xi, y, \gamma)$ to $J(\xi', y, \gamma)$.

Here is more detail on how the extended group representation of Definition 5.6 is constructed. First, the element $e(t/2)\delta_0$ is a generator for the extended Cartan:

$$({}^{\delta_0}H)^{\theta_\xi} = \langle e(t/2)\delta_0, H^{\theta_\xi}\rangle. \tag{37a}$$

The one-dimensional module $\mathbb{C}_{y,\gamma}$ extends to a one-dimensional $(\mathfrak{h}, ({}^{\delta_0}H)^{\theta_\xi})$-module by declaring

$$e(t/2)\delta_0 \text{ acts by the scalar } z(\lambda, \tau, \ell, t). \tag{37b}$$

Cohomological induction from this one-dimensional representation provides an extension of $I(\xi, y, \gamma)$ to a $(\mathfrak{g}, {}^{\delta_0}K_\xi)$-module $I_z(\lambda, \tau, \ell, t)$, and then $J_z(\lambda, \tau, \ell, t)$ is its unique irreducible quotient. Of course exactly the same words describe $J_z(\lambda', \tau', \ell', t')$.

So how do we decide whether these two modules are equivalent? According to (32), we can find $u \in X_*(H)$ so that

$$\begin{aligned} \ell' &= \ell + (\theta_x + 1)u \\ f' &= f + (\theta_x + 1)(\delta_0 - 1)u. \end{aligned} \tag{37c}$$

It follows that

$$e(u/2) \cdot \xi \cdot e(-u/2) = \xi'. \tag{37d}$$

If we define

$$t_2 = t + (\delta_0 - 1)u, \tag{37e}$$

then (ℓ', t_2) is another choice of representative for x as in (26), and in fact conjugate to (ℓ, t) by $e(u/2)$:

$$e(u/2) \cdot e(t/2)\delta_0 \cdot e(-u/2) = e(t_2/2)\delta_0. \tag{37f}$$

Consequently,

$$i =_{\text{def}} t' - t_2 \in X_*^{-\theta_x}, \qquad t' = t + (\delta_0 - 1)u + i. \tag{37g}$$

In exactly the same way, we find

$$\begin{aligned}
\lambda' &= \lambda + ({}^\vee\theta_y + 1)\omega \qquad \text{(some } \omega \in X^*(H))\\
\tau' &= \tau + ({}^\vee\delta_0 - 1)\omega + \iota \qquad \text{(some } \iota \in X^*(H)^{-\theta_y})\\
\phi' &= \phi + ({}^\vee\delta_0 - 1)({}^\vee\theta_y + 1)\omega.
\end{aligned} \tag{37h}$$

Proposition 6.2. *Suppose* $E = (\lambda, \tau, \ell, t)$ *and* $E' = (\lambda', \ell', \tau', t')$ *are extended parameters for* (x, y)*. Then*

$$\operatorname{sgn}(E, E') = (-1)^{\langle (1+{}^\vee\delta_0)\tau, u\rangle}(-1)^{\langle \iota, t'\rangle}.$$

Here u *and* ι *are defined in* (37c)*,* (37g)*, and* (37h)*.*

Proof. We change the parameter (λ, τ, ℓ, t) to $(\lambda', \ell', \tau', t)$ in three steps:

$$\begin{aligned}
E = (\lambda, \tau, \ell, t) \to F = (\lambda, \tau, \ell', t_2) \to&\\
G = (\lambda, \tau, \ell', t') \to E' =&(\lambda', \tau', \ell', t').
\end{aligned} \tag{38a}$$

In the first step we have conjugated by $e(u/2)$. It follows easily that the extended representations correspond if and only if the scalars chosen for the actions of $e(t/2)\delta_0$ and $e(t_2/2)\delta_0$ agree. That is,

$$\operatorname{sgn}(E, F) = z(E)/z(F). \tag{38b}$$

At the second step of (38a), we are keeping the group ${}^{\delta_0}K_{\xi'}$ the same, but changing the representative of the extended Cartan from $e(t_2/2)\delta_0$ to $e(t'/2)\delta_0$. This gives an equivalent extended parameter exactly if we multiply the scalar by

$$(-1)^{\langle \lambda, t'-t_2\rangle} = (-1)^{\langle \lambda, i\rangle}.$$

Therefore

$$\operatorname{sgn}(F, G) = \frac{z(F)}{z(G)}(-1)^{\langle \lambda, i\rangle}. \tag{38c}$$

Finally, in the last step of (38a) the group and the extended Cartan representative remain the same; all that may change is the scalar z. Therefore

$$\operatorname{sgn}(G, E') = z(G)/z(E'). \tag{38d}$$

Combining (38b)–(38d), we find

$$\operatorname{sgn}(E, E') = \frac{z(E)}{z(F)}\frac{z(F)}{z(G)}(-1)^{\langle \lambda, i\rangle}\frac{z(G)}{z(E')} = \frac{z(E)}{z(E')}(-1)^{\langle \lambda, i\rangle}. \tag{38e}$$

It remains to compute $z(E)/z(E')$. We do this in two steps. First of all we have from (29)

$$z(E)/z(G) = i^{\langle \tau, (1+\theta_x)(t-t')\rangle}(-1)^{\langle \lambda, t-t'\rangle}. \tag{38f}$$

With u and i given by (37g) this gives

$$z(E)/z(G) = i^{\langle \tau,(1+\theta_x)[(1-\delta_0)w+i]\rangle}(-1)^{\langle \lambda,(\delta_0-1)u+i\rangle} \tag{38g}$$

and a short computation using the identities gives

$$\begin{aligned} z(E)/z(G) &= (-1)^{\langle ({}^\vee\delta_0+{}^\vee\theta_y)\tau,u\rangle}(-1)^{\langle (1+{}^\vee\theta_y)\tau,u\rangle}(-1)^{\langle \lambda,i\rangle} \\ &= (-1)^{\langle (1+{}^\vee\delta_0)\tau,u\rangle}(-1)^{\langle \lambda,i\rangle}. \end{aligned} \tag{38h}$$

Next we compute

$$z(G)/z(E') = i^{\langle \tau-\tau',(1+\theta_x)t'\rangle}(-1)^{\langle \lambda-\lambda',t'\rangle}. \tag{38i}$$

Using (37h) this gives

$$\begin{aligned} z(G)/z(E') &= i^{\langle ({}^\vee\delta_0-1)\omega+\iota,(1+\theta_x)t'\rangle}(-1)^{\langle (1+{}^\vee\theta_y)\omega,t'\rangle} \\ &= (-1)^{\langle \omega,(1+\theta_x)t'\rangle}(-1)^{\langle \iota,t'\rangle}(-1)^{\langle \omega,(1+\theta_x)t'\rangle} \\ &= (-1)^{\langle \iota,t'\rangle}. \end{aligned} \tag{38j}$$

Multiplying (h) and (i) gives

$$z(E)/z(E') = (-1)^{\langle (1+{}^\vee\delta_0)\tau,u\rangle}(-1)^{\langle \lambda,i\rangle}(-1)^{\langle \iota,t'\rangle}. \tag{38k}$$

Multiplying both sides by $(-1)^{\langle \lambda,i\rangle\rangle}$ and using (38e) gives the result. □

6.1 Duality for extended parameters

We offer some remarks about duality in the sense of [15]. Define a group (called ${}^\vee G(e(\gamma))_0$ in (18d)):

$${}^\vee G(\gamma) = [\mathrm{Cent}_{{}^\vee G}(e(\gamma))]_0 \supset {}^\vee H, \tag{39a}$$

a connected reductive group with root system

$${}^\vee R(\gamma) = \{\alpha^\vee \in R^\vee \mid \langle \gamma,\alpha^\vee\rangle \in \mathbb{Z}\}, \tag{39b}$$

the integral roots for the infinitesimal character γ. The adjoint action of the representative

$${}^\vee\xi = e((\gamma-\lambda)/2){}^\vee\sigma_y{}^\vee\xi_0 \tag{39c}$$

defines an involutive automorphism of ${}^\vee G(\gamma)$, so

$${}^\vee K_{{}^\vee\xi} = \mathrm{Cent}_{{}^\vee G(\gamma)}({}^\vee\xi) \tag{39d}$$

is a symmetric subgroup of ${}^\vee G(\gamma)$. By symmetry, the parameter (y,x) defines an irreducible $({}^\vee\mathfrak{g}(\gamma),{}^\vee K_{{}^\vee\xi})$-module ${}^\vee J(x,y)$, with infinitesimal character g. (To be precise, we need to introduce a covering group related to the difference in ρ-shifts between ${}^\vee G$ and ${}^\vee G(\gamma)$, but we will overlook this technicality.) As in (24e), we find that

$$ {}^\vee h {}^\vee\delta_0 = e(\tau/2){}^\vee\delta_0 \tag{39e} $$

is a representative for an extended Cartan. As in Definition 5.2 we need to take a square root of

$$ \bar{\ell}(({}^\vee h {}^\vee\delta_0)^2) = (-1)^{\langle \ell, (1+{}^\vee\delta_0)\tau\rangle} \tag{39f} $$

which by (28) is precisely the sign $\epsilon(x,y)$ of Definition 5.2.

Therefore we may define an extended representation by making $e(\tau/2){}^\vee\delta_0$ act by any desired square root of ϵ. It turns out that duality dictates choosing a different square root than we did earlier; we choose ζ as in (35):

$$ \zeta(\lambda,\tau,\ell,t) = i^{\langle\tau,f\rangle}(-1)^{\langle\tau,\ell\rangle} = z(\lambda,\tau,\ell,t)(-1)^{\langle\lambda,t\rangle}(-1)^{\langle\ell,\tau\rangle}. \tag{39g} $$

Then define an extended representation ${}^\vee J_\zeta(\lambda,\tau,\ell,t)$ by

$$ e(\tau/2){}^\vee\delta_0 \mapsto \zeta(\lambda,\tau,\ell,t). \tag{39h} $$

The point of this choice of sign is that it makes the next result hold. Recall that if E, E' are parameters for (x,y), then $\mathrm{sgn}(E,E')$ is defined by the identity

$$ [J_z(E)] = \mathrm{sgn}(E,E')[J_z(E')], $$

and a formula for it is given in Proposition 6.2.

Proposition 6.3. *Suppose E, E' are extended parameters for (x,y). Then*

$$ [{}^\vee J_\zeta(E)] = \mathrm{sgn}(E,E')[{}^\vee J_\zeta(E')], $$

where $\mathrm{sgn}(E,E')$ *is defined in Definition 6.1. Equivalently,*

$$ J_z(E) \simeq J_z(E') \text{ if and only if } {}^\vee J_\zeta(E) \simeq {}^\vee J_\zeta(E'). \tag{40} $$

The proof is identical to that of Proposition 6.2. What matters for us, and what is by no means automatic, is that the sign is the same as the sign in Definition 6.1. We deduce

Corollary 6.4. *In the setting* (26)*, there is a natural bijection from δ_0-fixed extended representations (of strong real forms of infinitesimal cocharacter g) of G, of infinitesimal character γ; to ${}^\vee\delta_0$-fixed representations of (strong real forms of infinitesimal cocharacter γ) of ${}^\vee G(\gamma)$, of infinitesimal character g. The bijection sends $J_z(\lambda,\tau,\ell,t)$ to ${}^\vee J_\zeta(\lambda,\tau,\ell,t)$.*

The fact that this map is well defined on equivalence classes is precisely (40). In Section 11 we use this to extend the duality of [15] to the twisted setting.

The formulations of these results are designed to allow a theoretical analysis of all possible parameters for extended representations. For computational purposes, one may simply want to ask when two given parameters are equivalent. To answer that question using the results above requires calculating elements u and ω by solving their defining equations (37c) and (37h). This is not enormously difficult, but it is not necessary. We therefore conclude this section with a simpler formula for $\mathrm{sgn}(E, E')$.

Proposition 6.5. *Suppose E and E' are extended parameters for (x, y). Then*

$$\begin{aligned}\mathrm{sgn}(E,E') &= i^{\langle({}^\vee\delta_0-1)\lambda,t'-t\rangle+\langle\tau'-\tau,(\delta_0-1)\ell'\rangle}(-1)^{\langle\tau,\ell'-\ell\rangle+\langle\lambda'-\lambda,t'\rangle+\langle\tau,t'-t\rangle}\\ &= i^{\langle\tau',(\delta_0-1)\ell'\rangle-\langle\tau,(\delta_0-1)\ell\rangle}(-1)^{\langle\tau,\ell'-\ell\rangle}(-1)^{\langle\lambda'-\lambda,t'\rangle}.\end{aligned}$$

Here the two expressions on the right are automatically equal, and the powers of i appearing are automatically even.

The proof is similar to the proofs of Propositions 6.2 and 6.3. We omit the details.

7 Hecke algebra action

Our goal is to compute the Hecke algebra action defined in [10]. We begin by summarizing the definition of this Hecke algebra module. We then explain what extra information is needed, beyond the formulas of [10, Sections 7.5–7.7], to carry out the computation.

In Sections 7 through 9 we consider the case of integral infinitesimal character. In Section 10 we discuss the modification necessary to treat the general case.

We start with our group G and a pair of commuting involutions δ_0, ξ_0 as in Section 2. Fix a regular, integral infinitesimal character $\gamma \in \mathfrak{h}^*$. As always we assume γ is integrally dominant as in Definition 3.5; since γ is integral this means γ is dominant: ${}^\vee\alpha(\gamma) \in \mathbb{Z}_{>0}$ for all $\alpha > 0$. Let $\mathcal{H}$ be the twisted Hecke algebra of [10, Section 4], and set $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Fix a strong involution ξ inner to ξ_0, and set $K = K_\xi$. Associated to γ is an $\mathcal{H}$-module M, defined in [10, Section 2.3]. In our setting this is a quotient of the Grothendieck group over $\mathcal{A}$ of $(\mathfrak{g}, {}^{\delta_0}K)$-modules with infinitesimal character γ. Write $[X]$ for the image in M of a $(\mathfrak{g}, {}^{\delta_0}K)$-module X. Let χ be the non-trivial extension of the trivial representation of a one-dimensional $(\mathfrak{g}, {}^{\delta_0}K)$-module. In M we have the relation

$$[X] + [X \otimes \chi] \equiv 0.$$

Therefore M has a basis consisting of one extension to $(\mathfrak{g}, {}^{\delta_0}K)$ of each irreducible δ_0-fixed $(\mathfrak{g}, K)$-module with infinitesimal character γ. Furthermore if J is irreducible, and is not the extension of an irreducible $(\mathfrak{g}, K)$-module, then $[J] \equiv 0$.

Associated to a δ_0-orbit κ of simple roots is a generator T_κ of $\mathcal{H}$. Suppose I is a standard, δ_0-fixed $(\mathfrak{g}, K)$-module with infinitesimal character γ, and $\widetilde{I}$ is an extension of I to a $(\mathfrak{g}, {}^{\delta_0}K)$-module. Then formulas for $T_\kappa([\widetilde{I}])$ given in [10, Sections

7.5–7.7] are of the following form. There is a set $\{I_i \mid 1 \leq i \leq n\}$ (with $n \leq 3$) of standard, δ_0-fixed $(\mathfrak{g}, K)$-modules, such that the appropriate formula

$$T_\kappa([\widetilde{I}]) = \sum_i a_i [\widetilde{I}_i] \tag{41}$$

of [10] holds *for some choices of extension of each* I_i *to a* $(\mathfrak{g}, {}^{\delta_0}K)$*-module* $\widetilde{I}_i$. If we choose each extension $\widetilde{I}_i$ arbitrarily, then (41) holds with a factor of ± 1 in front of each term on the right.

It is natural to ask if it is possible to choose the $\widetilde{I}_i$ uniformly, so that the formulas (41) hold for all I and κ. The fact that in the `2i12` and `2r21` cases there is a term with a negative sign is a hint that this might not be the case, and it turns out not to be possible in general.

Instead, we carry over the Hecke module structure to our extended parameters, and compute the Hecke operators in this setting, keeping the extra information of which extensions (i.e., signs) appear in the formulas. This is straightforward except when κ is of type `2i12f, 2r21f, 2Ci` or `2Cr`.

Definition 7.1. Let $\mathcal{M}$ be the $\mathcal{A}$-module spanned by the extended parameters of infinitesimal character γ, modulo the relation

$$[E] \equiv \operatorname{sgn}(E, E')[E'].$$

By (36d) the map $[J_z(E)] \to [E]$ is a well-defined $\mathcal{A}$-module isomorphism. Using this we carry over the $\mathcal{H}$-module structure on M to define $\mathcal{M}$ as an $\mathcal{H}$-module.

To interpret the formulas of [10, Sections 7.5–7.7] in terms of $\mathcal{M}$ we need the notion of Cayley transform (defined only for certain particular κ) and cross action (defined for *every* κ) of extended parameters (defined in that reference). The rows of Tables 2–4 corresponding to Cayley transforms are labeled *Cay*, and those for cross action *crx*.

In addition, when κ is of type `2i12, 2r21, 2Ci` or `2Cr`, the formulas in [10] make use of one more transform, given (on the level of parameters for G) by the cross action of just *one* of the two simple roots comprising κ. On most parameters, this cross action will not give a δ_0-fixed parameter; like the Cayley transforms, the definition makes sense only when κ is of one of these four special types. The corresponding rows of Table 3 are labeled *cr1x*.

These formulas are given in Tables 2–4. Except in the cases noted above, this gives the formulas for the Hecke algebra action (see Proposition 7.2).

Here are some notes for interpreting the tables. Always we start with a δ_0-fixed representation of (${}^\vee\delta_0$-fixed) infinitesimal character γ, for a strong real form of δ_0-fixed infinitesimal cocharacter g, with `atlas` parameter (x, y). Let (λ, τ, ℓ, t) be an extended parameter for (x, y) (Definition 5.4).

We also fix a ${}^\vee\delta_0$-orbit κ on the set of simple roots, consisting of either

$$\begin{array}{lll} \text{one root } \{\alpha = {}^\vee\delta_0(\alpha)\} & & \text{(type 1); or} \\ \text{two roots } \{\alpha, \beta = {}^\vee\delta_0(\alpha)\}, & \langle\alpha, \beta^\vee\rangle = 0 & \text{(type 2); or} \\ \text{two roots } \{\alpha, \beta = {}^\vee\delta_0(\alpha)\}, & \langle\alpha, \beta^\vee\rangle = -1 & \text{(type 3).} \end{array} \tag{42a}$$

We will sometimes write

$$\kappa =_{\text{def}} \alpha + \beta \in X^*, \qquad \kappa^\vee = \alpha^\vee + \beta^\vee \in X_* \tag{42b}$$

in types 2 and 3. (The weight κ is a root in type 3, but not in type 2.) Let

$$w_\kappa = \begin{cases} s_\alpha & \text{type 1} \\ s_\alpha s_\beta & \text{type 2} \\ s_\alpha s_\beta s_\alpha = s_\kappa & \text{type 3.} \end{cases} \tag{42c}$$

Then W^{δ_0} is a Coxeter group with these elements as Coxeter generators.

We will write (x_1, y_1) for the `atlas` parameters defining (one of) the other δ_0-fixed representations appearing in the action of the Hecke algebra generator T_κ on (x, y), given by a (possibly iterated) cross action or Cayley transform. The point of the tables is to calculate new extended parameters, denoted $E_1 = (\lambda_1, \tau_1, \ell_1, t_1)$, for (x_1, y_1) in terms of $E = (\lambda, \tau, \ell, t)$.

Write $w_\kappa \times E$ for the cross action on E (described in the crx rows of Tables 2–4). Write

$$w_\kappa \times_1 E \qquad (\kappa \text{ of type } \mathtt{2i12} \text{ or } \mathtt{2r21}) \tag{42d}$$

for the element extending $s_\alpha \times (\lambda, \ell)$ defined in the cr1x rows of Table 3. Finally, write

$$c_\kappa(E) = E_\kappa \quad \text{or} \quad c_\kappa(E) = \{E_\kappa, E'_\kappa\} \tag{42e}$$

for the (possibly multi-valued) Cayley transform defined by the Cay rows of Tables 2–4.

We will write

$$\gamma_\alpha =_{\text{def}} \langle\gamma, \alpha^\vee\rangle, \quad g_\alpha =_{\text{def}} \langle\alpha, g\rangle, \tag{42f}$$

and similarly for λ and ℓ; these quantities are all integers. The δ_0-fixed requirement means that

$$\gamma_\alpha = \gamma_\beta, \quad g_\alpha = g_\beta \qquad (\text{types } \mathtt{2} \text{ and } \mathtt{3}). \tag{42g}$$

The δ_0-fixed requirement on λ and ℓ is more subtle, and with the details depending on the case. For example, we have

$$\lambda_\alpha + \lambda_\beta = 2(\gamma_\alpha - 1), \quad \ell_\alpha = \ell_\beta \qquad (\text{type } \mathtt{2Ci}). \tag{42h}$$

A few (but not many) such conditions are recorded in the notes column.

The notes column of the tables includes additional notation peculiar to some cases. For example, the case `1i1` corresponds to a discrete series in a block for A_1 with two discrete series and just one principal series. This turns out to mean that the root α must be trivial on the fixed points $H^{\theta_{x_1}}$ for the more split Cartan; and this in turn is equivalent to the existence of $\sigma \in X^*(H)$ so that

$$\alpha = (1 + {}^\vee\theta_{y_1})\sigma. \tag{42i}$$

That is the meaning of the note in the `1i1` row; the weight σ (which one needs to find by solving (42i) to implement the algorithm) appears in the formula for τ_1.

The terminology here is more compact than that of [10]. See Table 1.

Proposition 7.2. *Suppose (x, y) is a δ_0-fixed parameter, and E is an extended parameter for (x, y). Let κ be a δ_0-orbit of simple roots.*

Suppose κ is not of type `2i12, 2r21, 2Ci` or `2Cr`. Then the formulas for the action of the Hecke operator T_κ from [10] apply, using the Cayley transforms and cross actions from Tables 2–4, to give a formula for $T_\kappa([E])$.

This is a direct translation of the calculations of [10, Sections 7.5–7.7] to our setting. We treat the excluded cases in the next two sections.

Example 7.3. Suppose κ is a *two-imaginary noncompact type I-I ascent* for (x, y) (in the terminogy of [10]), i.e., of type `2i11` (in our terminology). Suppose E_1 is an extended parameter for (x, y), and set $E_2 = w_\kappa \times E_1$, and set $E' = c_\kappa(E_1)$, as defined by Table 3. Then formula [10, (7.6)(e′)] gives:

$$\begin{aligned} T_\kappa([E_1]) &= [E_2] + [E'] \\ T_\kappa([E_2]) &= [E_1] + [E'] \\ T_\kappa([E']) &= (q-1)([E_1] + [E_2]) + (q-2)[E']. \end{aligned}$$

8 The `2i12` case

If κ is of type `2i12f` or `2r21f`, then the formulas for the Hecke operator T_κ do not carry over directly from [10, (7.6)(i′′) and (j′′)]. We start with a special case.

Lemma 8.1. *Suppose κ is of type `2i12f` for $E_0 = (\lambda, \tau, \ell, t)$. Assume*

$$\begin{aligned} \tau_\alpha &= \tau_\beta = 0 \\ t_\alpha &= t_\beta = 0 \\ g_\alpha - \ell_\alpha &= g_\beta - \ell_\beta = 1 \\ \gamma_\alpha - \lambda_\alpha &= \gamma_\beta - \lambda_\beta = 1. \end{aligned} \tag{43}$$

Let $E_0' = w_\kappa \times_1 E_0$ as given by Table 3. (Recall that this is a certain extension of the parameter $s_\alpha \times (\lambda, \ell)$ for G. The Cayley transform $c_\kappa(E_0)$ is double-valued; write $c_\kappa(E_0) = \{F_0, F_0'\}$, where the parameter for F_0 is (λ, τ, ℓ, t), and $F_0' = w_\kappa \times_1 F_0$.

The action of T_κ on the space spanned by E_0, E_0', F_0, F_0' is

$$\begin{aligned} T_\kappa(E_0) &= E_0 + F_0 + F_0' \\ T_\kappa(E_0') &= E_0' + F_0 - F_0' \\ T_\kappa(F_0) &= (q^2-1)(E_0+E_0') + (q^2-2)F_0 \\ T_\kappa(F_0') &= (q^2-1)(E_0-E_0') + (q^2-2)F_0'. \end{aligned} \tag{44}$$

Explicitly the extended parameters are:

$$\begin{array}{ll} E_0 : (\lambda,\tau,\ell,t) & E_0' : (\lambda,\tau,\ell+\alpha^\vee,t-s) \\ F_0 : (\lambda,\tau,\ell,t) & F_0' : (\lambda+\alpha,\tau-\sigma,\ell,t) \end{array} \tag{45}$$

with σ and s given in Table 3.

When the parameters are in this form, this is simply a direct translation of the proof of [10, (7.6)(i′′)].

Lemma 8.2. *Suppose E is an extended parameter for (x,y), and κ is of type* `2i12f` *for E. Set $E' = w_\kappa \times_1 E$. Write $c_\kappa(E) = c_\kappa(E') = \{F,F'\}$. Possibly after switching E and E', and possibly also switching F and F', we can find E_0, E_0', F_0, F_0' as in the previous lemma, such that E and E_0 are extensions of the same parameter, and similarly $(E',E_0'), (F,F_0)$ and (F',F_0').*

Proof. Write $E = (\lambda,\tau,\ell,t)$, so $E' = (\lambda,\tau,\ell+\alpha,t-s)$. After replacing τ with a different solution of its defining equation:

$$\tau \to \tau + \tau_\beta\sigma + \frac{1}{2}(\tau_\alpha+\tau_\beta)\alpha$$

where $\alpha-\beta = (1+{}^\vee\theta_{y_1})\sigma$, we can assume $\tau_\alpha = \tau_\beta = 0$.

Since $2\alpha^\vee, 2\beta^\vee$ and $\alpha^\vee-\beta^\vee$ are all in $(1+\theta_x)X_*$, $a\alpha^\vee+b\beta^\vee$ is in $(1+\theta_x)X_*$ provided $a+b\in 2\mathbb{Z}$. By adding such a term to ℓ we can arrange that $g_\alpha-\ell_\alpha-1=0$ and $g_\beta-\ell_\beta-1=0$ or 2. Make the corresponding change $t\to t+\frac{1}{2}(b-a)(\alpha^\vee-\beta^\vee)$. If $g_\beta-\ell_\beta-1=2$, replace E with $E'=w_\kappa\times_1 E$, and now we have

$$g_\alpha-\ell_\alpha-1 = g_\beta-\ell_\beta-1 = 0.$$

Since $g_\alpha = g_\beta$ this implies $\ell_\alpha = \ell_\beta$. Then Conditions (a) and (d) of Definition 5.4 imply $\lambda_\alpha-\gamma_\alpha-1 = \lambda_\beta-\gamma_\beta-1 = 0$ and $t_\alpha = t_\beta = 0$. Table 3 then says that $F = (\lambda,\tau,\ell,t)$ is one of the two Cayley transforms of E, and that our parameters now have the form (45). □

Proposition 8.3. *In the setting of the previous lemma we have*

$$\begin{aligned} T_\kappa(E) &= E + \mathrm{sgn}(E,E_0)(\mathrm{sgn}(F,F_0)F + \mathrm{sgn}(F',F_0')F') \\ T_\kappa(E') &= E' + \mathrm{sgn}(E',E_0')(\mathrm{sgn}(F,F_0)F - \mathrm{sgn}(F',F_0')F') \\ T_\kappa(F) &= (q^2-1)\mathrm{sgn}(F,F_0)(\mathrm{sgn}(E,E_0)E + \mathrm{sgn}(E',E_0')E') + (q^2-2)F \\ T_\kappa(F') &= (q^2-1)\mathrm{sgn}(F',F_0')(\mathrm{sgn}(E,E_0)E - \mathrm{sgn}(E',E_0')E') + (q^2-2)F'. \end{aligned}$$

This formula is independent of the choice of E_0, E_0', F_0, F_0'.

This is immediate.

9 The 2Ci case

Now we describe the Hecke algebra action in the 2Ci case. So fix a type 2 root $\kappa = \{\alpha, \beta\}$, and an extended parameter

$$E = (\lambda,\ \tau,\ \ell,\ t) \tag{46a}$$

as in 5.4. Assume that κ is of type 2Ci for E: that is, that α and β are complex roots interchanged by

$$\theta_x = \mathrm{Ad}(e((g-\ell)/2)\sigma_w\xi_0). \tag{46b}$$

This means in turn that

$$w\xi_0\alpha = \beta, \qquad w\xi_0\beta = \alpha. \tag{46c}$$

Proposition 12.2 says that

$$\sigma_w\xi_0 X_\alpha = X_\beta, \qquad \sigma_w\xi_0 X_\beta = X_\alpha, \tag{46d}$$

and therefore that

$$\theta_x(X_\alpha) = (-1)^{g_\beta - \ell_\beta} X_\beta, \quad \theta_x(X_\beta) = (-1)^{g_\alpha - \ell_\alpha} X_\beta. \tag{46e}$$

The requirement (27) implies that

$$\gamma_\alpha = \gamma_\beta, \qquad g_\alpha = g_\beta. \tag{46f}$$

Similarly, the requirements for an extended parameter to be δ_0-fixed imply among other things that

$$\lambda_\alpha + \lambda_\beta = 2(\gamma_\alpha - 1), \quad \lambda_\alpha - \lambda_\beta = \tau_\beta - \tau_\alpha, \quad \ell_\alpha = \ell_\beta, \quad t_\alpha = -t_\beta. \tag{46g}$$

In particular, we can define a sign

$$\epsilon = \epsilon(E) = (-1)^{g_\alpha - \ell_\alpha} = (-1)^{g_\beta - \ell_\beta}. \tag{46h}$$

Writing

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \tag{46i}$$

for the eigenspace decomposition under θ_x, we get from (46e)

$$X_\alpha + \epsilon X_\beta =_{\text{def}} X_{\mathfrak{k}} \in \mathfrak{k}, \qquad X_\alpha - \epsilon X_\beta =_{\text{def}} X_{\mathfrak{s}} \in \mathfrak{s}, \tag{46j}$$

and also

$$\theta_x(\sigma_\alpha) = \sigma_\beta^\epsilon. \tag{46k}$$

The Weyl group element $s_\alpha s_\beta$ is represented by

$$\sigma_E = \sigma_\alpha \sigma_\beta^\epsilon \in G^{\theta_x} = K. \tag{46l}$$

Finally, the extended group ${}^{\delta_0}K$ is generated by K and the element

$$h = e(t/2)\delta_0 \tag{46m}$$

(Definition 5.4(e)).

Table 3 constructs from E a second extended parameter:

$$\begin{aligned} E_1 &= (\lambda_1, \tau_1, \ell_1, t_1) \\ &= (s_\alpha\lambda + (\gamma_\alpha - 1)\alpha, s_\alpha\tau - (\lambda_\alpha - \gamma_\alpha + 1)\alpha, \\ &\quad\; s_\alpha\ell + (g_\alpha - 1)\alpha^\vee, s_\alpha t + (\ell_\alpha - g_\alpha + 1)\alpha^\vee). \end{aligned} \tag{46n}$$

The root κ is of type `2Cr` for the parameter E_1. The element ℓ_1 is chosen so that the corresponding Cartan involution (on all of G, not just H) is

$$\theta_{x_1} = \sigma_\alpha^{-1}\theta_x\sigma_\alpha. \tag{46o}$$

Proposition 9.1. *Suppose we are in the setting* (46)*.*

1. *Applying the formula in Table 3 to the* `2Cr` *parameter E_1 gives exactly the same parameter E with which we started.*
2. *The action of the Hecke algebra generator T_κ ([10, 7.6(c″)]) is*

$$T_\kappa(E) = qE + (-1)^{[(\tau_\alpha+\tau_\beta)/2](g_\alpha-\ell_\alpha-1)}(q+1)E_1.$$

Here the sign may be regarded as specifying a renormalization of E_1 (whose existence is asserted in [10]).

3. *The corresponding formula for the case* `2Cr` *is*

$$T_\kappa(E_1) = (q^2-q-1)E_1 + (-1)^{(\gamma_{1,\alpha}-\lambda_{1,\alpha}+\tau_{1,\alpha}-1)[(t_{1,\beta}-t_{1,\alpha})/2]}(q^2-q)E.$$

The sign is exactly the same as the one for $T_\kappa(E)$, written in terms of the parameter E_1.

Proof. The first assertion can be verified by applying the formulas for passing from `2Ci` to `2Cr` and from `2Cr` to `2Ci` in succession, then simplifying; we omit the details.

For the second assertion, we need to understand representation-theoretically the relationship between the extended parameters E and E_1, and how this relates to the Hecke algebra action. For this question it is easiest to think of $(\mathfrak{g}, K)$-modules with a fixed K; that is, to conjugate θ_{x_1} back to θ_x, and to correspondingly change E_1 into a parameter

$$\begin{aligned} E_2 &= (\lambda_2, \tau_2, \ell_2, t_2) \\ &= (\lambda - (\gamma_\alpha - 1)\alpha, \tau + (\lambda_\alpha - \gamma_\alpha + 1)\alpha, \\ &\qquad \ell - (g_\alpha - 1)\alpha^\vee, t - (\ell_\alpha - g_\alpha + 1)\alpha^\vee) \end{aligned} \tag{47a}$$

related to the Borel subgroup

$$B' = \sigma_\alpha B \sigma_\alpha^{-1}. \tag{47b}$$

(The `atlas` decision to prefer E_1 to E_2 is just a bookkeeping convenience. Everything about representation theory, and also most things about perverse sheaves, are calculated with a fixed Cartan involution, and so refer to the relationship between E and E_2. The `atlas` formulas for Hecke algebra actions index bases by E_1 rather than E_2, so we will occasionally mention E_1 below; but mostly we will be concerned about E and E_2.)

The distinguished automorphism corresponding to δ_0 for B' is

$$\delta_0' = \sigma_\alpha \delta_0 \sigma_\alpha^{-1} = \sigma_\alpha \sigma_\beta^{-1} \delta_0. \tag{47c}$$

The generator for the extended Cartan defined by E_2 is

$$h_2 = e(t_2/2)\delta_0' = e(t/2) m_\alpha^{\ell_\alpha - g_\alpha + 1} \sigma_\alpha \sigma_\beta^{-1} \delta_0 = e(t/2)\sigma_E^{-\epsilon}\delta_0 \tag{47d}$$

(with $\sigma_E \in K$ as in (46l). Now we can start to talk about representation theory: that is, about $(\mathfrak{g}, K)$-modules M and their extensions to $(\mathfrak{g}, {}^{\delta_0}K)$-modules M'. Write

$$P = LU \supset B,\ B' \tag{47e}$$

for the parabolic subgroup with L generated by H and the simple roots α and β. Then

$$M_j = H_j(\mathfrak{u}, M), \qquad M_j' = H_j(\mathfrak{u}, M') \tag{47f}$$

are $(\mathfrak{l}, L \cap K)$- and $(\mathfrak{l}, {}^{\delta_0}(L \cap K))$-modules respectively; and the relationship between representations and parameters (which uses $\mathfrak{n}$-homology) factors through this construction by means of the Hochschild–Serre spectral sequence. In this way (omitting details) one can reduce the questions we are studying to the case

$$G = L, \quad R = \{\pm\alpha, \pm\beta\}. \tag{47g}$$

In the setting (47g), here is what the representation theory looks like. The group L is locally $SL(2) \times SL(2)$, and K is approximately a "diagonal" copy of $SL(2)$. (More precisely, the "diagonal" copy is

$$SL(2)_K = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & \epsilon b \\ \epsilon c & d \end{pmatrix} \right) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \right\}, \tag{47h}$$

with ϵ as in (46h). Furthermore K, and even its intersection with the derived group of L, may be disconnected.

Attached to E is an irreducible principal series $(\mathfrak{g}, {}^{\delta_0}K)$-module $I(E)$. The restriction of $I(E)$ to ${}^{\delta_0}K$ is

$$I(E) = \mathrm{Ind}_{{}^{\delta_0}(H\cap K)}^{{}^{\delta_0}K}(\Lambda_E + 2\rho_n). \tag{47i}$$

Here Λ_E is the character of $H \cap K$ defined by the first term λ in E, extended to ${}^{\delta_0}(H \cap K)$ by making h (from (46m)) act by (29). The twist $2\rho_n$ is the character by which ${}^{\delta_0}(H \cap K)$ acts on (the top exterior power of) $\mathfrak{n} \cap \mathfrak{s}$; that is, on the vector $X_{\mathfrak{s}}$ from (46j).

The reason for the last twist is that for fundamental series modules M, the character of $H \cap K$ in the parameter is a weight on $H_{\dim(\mathfrak{n}^{\mathrm{op}}\cap\mathfrak{s})}(\mathfrak{n}^{\mathrm{op}}, M)$, specifically appearing in the image of a natural map

$$H_0(\mathfrak{n}^{\mathrm{op}} \cap \mathfrak{k}, M) \otimes \bigwedge^{\dim(\mathfrak{n}^{\mathrm{op}}\cap\mathfrak{s})} (\mathfrak{n}^{\mathrm{op}} \cap \mathfrak{s}) \to H_{\dim(\mathfrak{n}^{\mathrm{op}}\cap\mathfrak{s})}(\mathfrak{n}^{\mathrm{op}}, M).$$

The conclusion is that *the weight of $H \cap K$ on the parameter is equal to the $\mathfrak{n} \cap \mathfrak{k}$-highest weight of the lowest K-type, minus $2\rho_n$*. The coroot for K is

$$\alpha^\vee + \beta^\vee, \tag{47j}$$

which acts on $X_{\mathfrak{s}}$ by 2. The dimension of the lowest K-type is therefore

$$\lambda_\alpha + \lambda_\beta + 2 + 1 = 2\gamma_\alpha + 1; \tag{47k}$$

the 1 comes from the ρ-shift in the Weyl dimension formula, and we have used (46g) to convert λ to γ. In particular, we find that

$$I(E)|_{SL(2)_K} = \text{sum of irreducibles of dimensions } 2\gamma_\alpha + 1,\, 2\gamma_\alpha + 3, \ldots. \tag{47l}$$

In the same fashion, attached to E_2 is a reducible principal series $(\mathfrak{g}, {}^{\delta_0}K)$-module $I(E_2)$. The restriction of $I(E_2)$ to ${}^{\delta_0}K$ is

$$I(E_2) = \mathrm{Ind}_{{}^{\delta_0}(H\cap K)}^{{}^{\delta_0}K}(\Lambda_{E_2}). \tag{47m}$$

The reason for the absence of a twist on Λ_{E_2} is that for principal series modules M_2 for quasisplit groups, the parameter appears as a weight on $H_0(\mathfrak{n}^{\mathrm{op}}, M_2)$; and for (almost) spherical representations, this weight space is precisely the image of the (almost) spherical vector. In particular,

$$I(E_2)|_{SL(2)_K} = \text{sum of irreducibles of dims } 1, 3, \ldots. \tag{47n}$$

The principal series representation $I(E_2)$ has a unique irreducible quotient representation $J(E_2)$:

$$\begin{aligned} J(E_2)|_{SL(2)_K} &= \text{sum of irreducibles of dims } 1, 3, \ldots, 2\gamma_\alpha - 1, \\ \dim J(E_2) &= \gamma_\alpha^2. \end{aligned} \tag{47o}$$

We get a short exact sequence

$$\begin{aligned} 0 \to & I(E') \to I(E_2) \to J(E_2) \to 0, \\ & I(E')|_{SL(2)_K} = \text{sum of irreducibles of dims } 2\gamma_\alpha + 1, 2\gamma_\alpha + 3, \ldots. \end{aligned} \tag{47p}$$

This extended parameter E' with $I(E')$ appearing as a composition factor of $I(E_2)$ is the one on which the Hecke algebra action gives E_1 (remember that this is essentially just another label for E_2) with positive coefficient. (This is a consequence of the Beilinson–Bernstein localization theory relating perverse sheaves to representations, and the perverse sheaf definition of the Hecke algebra action in [10].) So we need to understand the relationship between the extended parameters E and E'.

Because the spherical composition factor $J(E_2)$ is a unique quotient of $I(E_2)$, the spherical vector in $I(E_2)$ is cyclic. The action of $X_{\mathfrak{s}}$ carries highest weight vectors for K to highest weight vectors for K; so we deduce

$$\begin{aligned} & X_{\mathfrak{s}}^{\gamma_\alpha}(\text{spherical vector in } I(E_2)) \\ & \qquad = \text{highest weight vector for lowest } K\text{-type of } I(E'). \end{aligned} \tag{48a}$$

Because $\sigma_E \in SL(2)_K$ acts trivially on the (one-dimensional) lowest K-type of $J(E_2)$, the formula (47d) shows that

$$\Lambda_{E_2}(h_2) = \text{action of } e(t/2)\delta_0 \text{ on } J(E_2) \text{ lowest } K\text{-type}. \tag{48b}$$

It is easy to calculate

$$\mathrm{Ad}(e(t/2)\delta_0)(X_{\mathfrak{s}}) = -\epsilon(-1)^{t_\alpha} = (-1)^{g_\alpha - \ell_\alpha - 1 + t_\alpha}.$$

Combining (48b) with (48a), we find that the

$$\begin{aligned} & \text{action of } h = e(t/2)\delta_0 \text{ on } I(E') \text{ lowest } K\text{-type} \\ & = \Lambda_{E_2}(h_2)(-\epsilon)^{\gamma_\alpha}(-1)^{\gamma_\alpha t_\alpha} \\ & = \Lambda_{E_2}(h_2)(-1)^{\gamma_\alpha((g_\alpha - \ell_\alpha - 1) + t_\alpha)}. \end{aligned} \tag{48c}$$

Using the description of the parameter for E' given before (47j), we get

$$\Lambda_{E'}(h) = \Lambda_{E_2}(h_2)(-1)^{(\gamma_\alpha - 1)((g_\alpha - \ell_\alpha - 1) + t_\alpha)}. \tag{48d}$$

Now we compare this "desired" relationship between $\Lambda_{E'}(h)$ and $\Lambda_{E_2}(h_2)$ with the actual relationship between $\Lambda_E(h)$ and $\Lambda_{E_2}(h_2)$. We find (using (29) and the formulas in Table 3 for E_1)

$$\begin{aligned}\Lambda_E(h)\Lambda_{E_2}^{-1}(h_2) &= \Lambda_E(h)\Lambda_{E_1}^{-1}(h_1)\\ &= i^{\langle\tau,(\delta_0-1)\ell\rangle}(-1)^{\langle\lambda,t\rangle}\\ &\quad i^{-\langle\tau-[(\tau_\alpha+\tau_\beta)/2]]\alpha,(\delta_0-1)(\ell+(g_\alpha-\ell_\alpha-1)\alpha^\vee\rangle}\\ &\quad (-1)^{\langle(\lambda+(\gamma_\alpha-\lambda_\alpha-1)\alpha,t+(\ell_\alpha-g_\alpha-t_\alpha+1)\alpha^\vee\rangle}\\ &= i^{\langle[(\tau_\alpha+\tau_\beta)/2]\alpha,(\delta_0-1)(\ell+(g_\alpha-\ell_\alpha-1)\alpha^\vee\rangle}\\ &\quad i^{-\langle\tau,(\delta_0-1)((g_\alpha-\ell_\alpha-1)\alpha)\rangle}\\ &\quad (-1)^{\langle(\gamma_\alpha-\lambda_\alpha-1)\alpha,t+(\ell_\alpha-g_\alpha-t_\alpha+1)\alpha^\vee\rangle}\\ &\quad (-1)^{\langle\lambda,(\ell_\alpha-g_\alpha-t_\alpha+1)\alpha^\vee\rangle}.\end{aligned} \tag{48e}$$

There are four factors on the right. In the first,

$$\langle\alpha,(\delta_0-1)\ell\rangle = \ell_\alpha-\ell_\beta = 0$$

by (46g). In the third, $\langle\alpha,\alpha^\vee\rangle = 2$ contributes an even power of (-1), so can be dropped. We are left with

$$\begin{aligned}\Lambda_E(h)\Lambda_{E_2}^{-1}(h_2) =&\ i^{\langle[(\tau_\alpha+\tau_\beta)/2]\alpha,(\delta_0-1)((g_\alpha-\ell_\alpha-1)\alpha^\vee)\rangle} i^{-\langle\tau,(\delta_0-1)((g_\alpha-\ell_\alpha-1)\alpha)\rangle}\\ &\ (-1)^{\langle(\gamma_\alpha-\lambda_\alpha-1)\alpha,t\rangle}(-1)^{\langle\lambda,(\ell_\alpha-g_\alpha-t_\alpha+1)\alpha^\vee\rangle}\\ =&\ (-1)^{[(\tau_\alpha+\tau_\beta)/2](g_\alpha-\ell_\alpha-1)]}(-1)^{[(\tau_\alpha-\tau_\beta)/2](g_\alpha-\ell_\alpha-1)}\\ &\ (-1)^{(\gamma_\alpha-\lambda_\alpha-1)t_\alpha}(-1)^{\lambda_\alpha(\ell_\alpha-g_\alpha-t_\alpha+1)}\\ =&\ (-1)^{\tau_\alpha(g_\alpha-\ell_\alpha-1)}(-1)^{(\gamma_\alpha-\lambda_\alpha-1)t_\alpha}(-1)^{\lambda_\alpha(\ell_\alpha-g_\alpha-t_\alpha+1)}.\end{aligned} \tag{48f}$$

Splitting the last factor between the first two gives

$$\begin{aligned}\Lambda_E(h)\Lambda_{E_2}^{-1}(h_2) =&\ (-1)^{(\lambda_\alpha+\tau_\alpha)(g_\alpha-\ell_\alpha-1)}\\ &\ (-1)^{(\gamma_\alpha-1)t_\alpha}.\end{aligned} \tag{48g}$$

Now use the first two formulas from (46g) to write $\lambda_\alpha = (\gamma_\alpha-1)+(\tau_\beta-\tau_\alpha)/2$. We get

$$\begin{aligned}\Lambda_E(h)\Lambda_{E_2}^{-1}(h_2) &= (-1)^{[(\gamma_\alpha-1)+(\tau_\alpha+\tau_\beta)/2](g_\alpha-\ell_\alpha-1)}(-1)^{(\gamma_\alpha-1)t_\alpha}\\ &= (-1)^{(\gamma_\alpha-1)(g_\alpha-\ell_\alpha+t_\alpha-1)}(-1)^{[(\tau_\alpha+\tau_\beta)/2](g_\alpha-\ell_\alpha-1)}.\end{aligned} \tag{48h}$$

The first factor here is exactly the one from (48d), so we deduce

$$\Lambda_E(h) = \Lambda_{E'}(h)(-1)^{[(\tau_\alpha+\tau_\beta)/2][g_\alpha-\ell_\alpha-1]}. \tag{48i}$$

The sign on the right has to appear in front of the [10] Hecke algebra formula for the coefficient of E_1 in $T_\kappa E$. This proves the second assertion of the proposition. For the third, we just rewrite exactly the same formula in terms of the parameter E_1; by the first assertion of the proposition, the formulas in Table 3 tell us how to do that. We omit the algebraic details. □

We summarize the results of Sections 7–9.

Theorem 9.2. *If the infinitesimal character* γ *is integral, then the action of* $\mathcal{H}$ *on* $\mathcal{M}$ *(Definition 7.1) is given by Propositions 7.2, 8.3 and 9.1.*

10 Nonintegral infinitesimal character

Suppose the infinitesimal character γ is not necessarily integral. As always we assume it is integrally dominant (17). Set

$$^\vee R(\gamma) = \{^\vee\alpha \in {}^\vee R \mid \langle \gamma, {}^\vee\alpha \rangle \in \mathbb{Z}\} \tag{49a}$$

as in (39b), and set

$$\begin{aligned} R(\gamma) &= \{\alpha \in R \mid {}^\vee\alpha \in {}^\vee R(\gamma)\} \\ R(\gamma)^+ &= R^+ \cap R(\gamma). \end{aligned} \tag{49b}$$

We say $\alpha \in R$ is *integral* if $\alpha \in R(\gamma)$. We say an integral root is *simple* (respectively *integral-simple*) if it is simple for R^+ (respectively $R(\gamma)^+$).

The Weyl group $W(\gamma)$ of $R(\gamma)$ satisfies

$$W(\gamma) = \{w \in W \mid w\gamma - \gamma \in \mathbb{Z}R\}. \tag{49c}$$

We now assume ${}^\vee\delta_0(\gamma) = \gamma$ (see Lemma 4.1), so ${}^\vee\delta_0$ acts on $R(\gamma)$. Then ${}^\vee\delta_0$ preserves both the simple and integral-simple roots, so the notions of integral and integral-simple apply to a ${}^\vee\delta_0$-orbit $\kappa = \{\alpha, {}^\vee\delta_0(\alpha)\}$ of roots. Let $\mathcal{H}(\gamma)$ be the Hecke algebra of [10, (4.7)] applied to $(R(\gamma), \delta_0)$.

Let $\mathcal{M}_\gamma$ be the module of Definition 7.1. The construction of [10] gives a representation of $\mathcal{H}(\gamma)$ on $\mathcal{M}_\gamma$. (More precisely, the construction of [10] concerns geometry related by base change (to compare a base field of finite characteristic with $\mathbb{C}$) and Beilinson–Bernstein localization (to relate K-equivariant perverse sheaves to $(\mathfrak{g}, K)$-modules) to the module of Definition 7.1. In order to make a parallel identification in the case of nonintegral infinitesimal character, one needs a discussion like that in [1, Chapter 17]. We omit the details.)

Suppose κ is a ${}^\vee\delta_0$-orbit of roots that are integral (for γ) and simple (for G). Then the formulas of Tables 2–4 apply to give a formula for the action T_γ on $\mathcal{M}$. The technical issue we have to deal with here is what to do if κ is integral-simple (for γ) but not simple (for G).

Definition 10.1. Let $\mathcal{ID}$ be the set of integrally dominant elements of $\mathfrak{h}^*$:

$$\mathcal{ID} = \{\gamma \in \mathfrak{h}^* \mid \alpha \in R(\gamma)^+ \implies \langle \gamma, {}^\vee\alpha \rangle \geq 0\}.$$

If $\gamma \in \mathfrak{h}^*$, then γ is $W(\gamma)$-conjugate to a unique element of $\mathcal{ID}$. If $\gamma \in \mathcal{ID}$ and $w \in W$, let $w * \gamma$ be the unique element of $\mathcal{ID}$ which is $W(w\gamma)$ conjugate to $w\gamma$.

It is easy to see that $w * \gamma$ is the unique element satisfying

(a) $w * \gamma \in \mathcal{ID}$
(b) $w * \gamma$ is W-conjugate to γ
(c) $w * \gamma \in w\gamma + \mathbb{Z}R$.

Condition (c) is equivalent (in the presence of (a) and (b)) to

(c′) $w * \lambda = xw\lambda$ for some $x \in W(w\lambda)$.

Lemma 10.2. *The map $(w, \gamma) \to w * \gamma$ is an action of W on $\mathcal{ID}$. It satisfies:*

1. *$Stab_W(\gamma) = W(\gamma)$;*
2. *The W-orbit of γ under $*$ is in bijection with $W/W(\gamma)$;*
3. *$w * \gamma = xw\gamma$ for some $x \in W(w\gamma)$;*
4. *Suppose α is simple for R^+. Then*

$$s_\alpha * \gamma = \begin{cases} \gamma & \alpha \in R(\gamma) \\ s_\alpha(\gamma) & \alpha \notin R(\gamma). \end{cases}$$

Proof. If $x, y \in W$, then $(xy) * \lambda$ is the unique element satisfying conditions (a–c) above with respect to xy. On the other hand $x * (y * \lambda)$ obviously satisfies (a) and (b). Condition (c) holds as well:

$$\begin{aligned} x * (y * \gamma) &\in x(y * \gamma) + \mathbb{Z}R \\ &\in x(y\gamma + \mathbb{Z}R) + \mathbb{Z}R \quad = (xy)\gamma + \mathbb{Z}R. \end{aligned}$$

Assertions (1–3) are straightforward, and (4) is clear if α is integral for γ, so assume this is not the case. Obviously $s_\alpha(\gamma)$ satisfies the conditions (b) and (c) for $s_\alpha * \gamma$, and (a) follows from the fact that s_α permutes $R^+ - \{\alpha\}$. We leave the details to the reader. □

If γ is integral, the formulas for the cross action in Tables 2–4 define an action of W^{δ_0} on $\mathcal{M}$. With a small change the same holds in general. To indicate the role of γ, write $(\lambda, \tau, \ell, t, \gamma)$ for an extended parameter.

Definition 10.3. Suppose κ is a ${}^\vee\delta_0$-orbit of simple roots. Suppose $\gamma \in \mathcal{ID}$ and $(\lambda, \tau, \ell, t, \gamma)$ is an extended parameter. Use the formulas for the cross action of κ from Tables 2–4, applied to (λ, τ, ℓ, t), to define $(\lambda_1, \tau_1, \ell_1, t_1)$. Then define

$$w_\kappa \times (\lambda, \tau, \ell, t, \gamma) = (\lambda_1 + (w_\kappa * \gamma - \gamma), \tau_1, \ell_1, t_1, w_\kappa * \gamma).$$

Define the cross action of any element of W^{δ_0} by writing it as a product of w_κs.

If κ is integral then $w_\kappa * \gamma = \gamma$, and Definition 10.3 agrees with the definition of the cross action in Tables 2–4. The main point is that even if κ is not integral, the formula in Definition 10.3 gives a valid extended parameter. In particular the relation

$$(1-{}^\vee\theta_1)(\lambda_1)=(1-{}^\vee\theta_1)(\gamma-\rho)$$

holds exactly as in the integral case. What we need to know is that

$$(1-{}^\vee\theta_1)(\lambda_1+(w_\kappa*\gamma-\gamma))=(1-{}^\vee\theta_1)(w_\kappa*\gamma-\rho)$$

which follows immediately. Furthermore in the integral case $\lambda_1\in X^*$. In the non-integral case it follows readily from the definitions that $\lambda_1+(w_\kappa*\gamma-\gamma)\in X^*$ (even though this doesn't hold separately for λ_1 and $w_\kappa*\gamma-\gamma$).

Proposition 10.4. *Suppose $\gamma\in\mathcal{ID}$ is not necessarily integral. Then the action of $\mathcal{H}(\gamma)$ on $\mathcal{M}_\gamma$ is given by the formulas in Table 5, with the following changes.*

Suppose κ is integral-simple, $w\in W^{\delta_0}$, and these satisfy: $w\kappa$ is integral, simple (for R^+), and the Cayley transform $c_{w\kappa}(w\times E)$ is defined by Tables 2–4. Then define $c_\kappa(E)$ to be

$$c_\kappa(E)=w^{-1}\times c_{w\kappa}(w\times E)$$

where the cross action is that of Definition 10.3.

On the other hand, suppose $w\kappa$ is of type `2i11,2i12,2r11,2r12` *for $w\times E$, so $w\kappa\times(w\times E)$ is defined by Table 3. Define*

$$w_\kappa\times E=w^{-1}\times[w\kappa\times(w\times E)].$$

It is helpful to reformulate the action of W.

Lemma 10.5. *Suppose $E=(\lambda,\tau,\ell,t,\gamma)$ is an extended parameter, and $w\in W^{\delta_0}$. Then $w\times E=(\lambda',\tau',\ell',t',w*\gamma)$ where*

$$\begin{aligned}\lambda'&=w*\gamma-w(\gamma-\lambda)+(w\rho-\rho)-(w\rho_r(x)-\rho_r(wxw^{-1}))\\ \tau'&=w\tau-({}^\vee\delta_0-1)(w\rho_r(x)-\rho_r(wxw^{-1}))/2\\ \ell'&=g-w(g-\ell)+(w{}^\vee\rho-{}^\vee\rho)-(w\rho_r(y)-\rho_r(wyw^{-1})\\ t'&=wt-(\delta_0-1)(w\rho_r(y)-\rho_r(wyw^{-1}))/2.\end{aligned}$$

The proof is that these formulas agree with those Definition 10.3 when $w=w_\kappa$. We omit the details.

To apply the proposition we need the following lemma.

Lemma 10.6. *Suppose κ is a ${}^\vee\delta_0$-orbit of integral-simple roots. Then there exists $w\in W^{\delta_0}$ such that $w\kappa$ is simple, unless $\kappa=\{\alpha\}$ is of length 1 and (the simple factor of) G is locally isomorphic to $SL(2n+1,\mathbb{R})$.*

This follows from the facts that the "quotient" root system R/δ_0 [12] consisting of the restrictions of roots to H^{δ_0}, is a (possibly non-reduced) root system, with Weyl group W^{δ_0}; and in a reduced root system every root is conjugate to a simple root. The excluded case in the lemma is type A_{2n}, in which case R/δ_0 is the non-reduced system of type BC_n, and a δ_0-fixed root restricts to twice a root.

Extending Proposition 10.4 to this excluded case requires just a calculation in $SL(3,\mathbb{R})$, which we omit.

11 Duality

Definition 11.1. Let τ be the anti-automorphism of $\mathcal{H}$ given by

$$q\tau(T_\kappa) = -q^\ell T_\kappa^{-1} = -T_\kappa + (q^\ell - 1) \quad (\ell = \text{length}(\kappa)). \tag{50}$$

Suppose π is a representation of $\mathcal{H}$ on an $\mathcal{A}$-module V. The dual representation π^*, on $\mathrm{Hom}_{\mathcal{A}}(V,\mathcal{A})$ is given by

$$\pi^*(T_\kappa)(\lambda)(v) = \lambda(\pi(\tau(T_\kappa)v).$$

In the setting of Section 2 let $\mathcal{H}$ be the Hecke algebra for (G,δ_0) (see [10] and Section 7). Let ${}^\vee\mathcal{H}$ be the algebra given by the same construction applied to $({}^\vee G, {}^\vee\delta_0)$. If κ is a δ_0-orbit of simple roots for G, then ${}^\vee\kappa$ is a ${}^\vee\delta_0$-orbit of simple roots for ${}^\vee G$, and the map $T_\kappa \to T_{{}^\vee\kappa}$ induces a Hecke algebra isomorphism.

Fix (regular, rational) infinitesimal character γ and (regular, integral) infinitesimal character cocharacter g as in (26).

We now assume that γ is integral. Let $\mathcal{M}$ be the $\mathcal{H}$-module of Definition 7.1, applied to G, δ_0, and γ. Recall $\mathcal{M}$ is spanned by equivalence classes $[E]$, for E an extended parameter with infinitesimal character γ, and $[I_z(E)] \to [E]$ is an isomorphism of Hecke modules.

Let ${}^\vee\mathcal{M}$ be the ${}^\vee\mathcal{H}$-module obtained by applying the same construction to ${}^\vee G$, ${}^\vee\delta_0$ and g. If E is an extended parameter, write ${}^\vee E$ for the same parameter, viewed as an extended parameter for ${}^\vee G$. The map $[I_\zeta({}^\vee E)] \to [{}^\vee E]$ is an isomorphism of ${}^\vee\mathcal{H}$-modules. Write $[E]' \in \mathrm{Hom}_{\mathcal{A}}(V,\mathcal{A})$ for the dual basis vector.

Proposition 11.2. *The map $[E]' \to (-1)^{\mathit{length}(E)}[{}^\vee E]$ is an isomorphism of $\mathcal{H} \simeq {}^\vee\mathcal{H}$-modules.*

Proof. The statement is equivalent to the following assertion. For all κ, and extended parameters E, F:

$$\text{the coefficient of } [E] \text{ in } -T_\kappa([F]) + (u^{\ell(\kappa)} - 1)\mathrm{sgn}(E,F) \tag{51a}$$

is equal to

$$(-1)^{\ell(E)-\ell(F)} * \text{the coefficient of } [{}^\vee F] \text{ in } T_{{}^\vee\kappa}([{}^\vee E]). \tag{51b}$$

In (a) $\mathrm{sgn}(E,F)$ is defined to be 0 if E, F are not extensions of the same parameter.

Up to signs, all of these formulas can be read off easily from the formulas for the Hecke algebra action on parameters. See Table 5. The fact that the signs are correct is due to the symmetry of Table 5. This is best illustrated by an example.

Example 11.3. Suppose κ is type `1i1` for an extended parameter F. Then κ is also of type `1i1` for $w_\kappa \times F$. According to Table 5,

$$\text{the coefficient of } [w_\kappa \times F] \text{ in } -T_\kappa([F]) \text{ is } -1. \tag{52a}$$

We need to show this equals

$$-1(\text{the coefficient of } [{}^\vee F] \text{ in } T_{{}^\vee\kappa}([{}^\vee(w_\kappa \times F)]). \tag{52b}$$

From the same line in Table 5, applied to ${}^\vee G$, we know that

$$-(\text{the coefficient of } [{}^\vee F] \text{ in } T_{{}^\vee\kappa}([w_\kappa \times {}^\vee F]) = -1. \tag{52c}$$

So we need to know that

$$(w_\kappa \times F)^\vee \equiv w_\kappa \times {}^\vee F. \tag{52d}$$

This identity reflects a symmetry of the tables. Here $w_\kappa \times F$ is a cross action of type `1i1`, $w_\kappa \times {}^\vee F$ is of type `1r1`. Switching the roles of $\lambda \leftrightarrow \ell$, and $\tau \leftrightarrow t$ interchanges these two formulas.

The necessary symmetry holds for all Cayley transforms and cross actions; in Table 5 the dual operations are listed on the same line. This completes the proof of the proposition. □

12 Appendix

We collect a few technical results about the Tits group [13], which will be needed for our study of parameters for representations in Section 3. We continue with the notation of (11). For each simple root α, the pinning defines a canonical homomorphism

$$\phi_\alpha\colon (SL(2), \text{diag}) \to (G, H) \qquad d\phi_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_\alpha. \tag{53a}$$

Similarly,

$$\phi_{\alpha^\vee}\colon (SL(2), \text{diag}) \to ({}^\vee G, {}^\vee H) \qquad d\phi_{\alpha^\vee} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_{\alpha^\vee}. \tag{53b}$$

It is sometimes convenient to define also

$$H_\alpha = d\phi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_{-\alpha} = d\phi_\alpha \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \tag{53c}$$

the first element (because $\alpha(H_\alpha) = 2$) "is" the coroot $\alpha^\vee$. The second is a preferred root vector for $-\alpha$, characterized by the last of the three relations

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, X_{-\alpha}] = -2X_{-\alpha}, \quad [X_\alpha, X_{-\alpha}] = H_\alpha. \tag{53d}$$

In this way we get a distinguished representative

$$\begin{aligned} \sigma_\alpha &=_{\text{def}} \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp(\frac{\pi}{2}(X_\alpha - X_{-\alpha})) \\ \sigma_\alpha^2 &= m_\alpha =_{\text{def}} \alpha^\vee(-1) \end{aligned} \tag{53e}$$

for the simple reflection s_α. These representatives satisfy the braid relations (see [13]) and therefore define distinguished representatives

$$\sigma_w =_{\text{def}} \sigma_{\alpha_1}\sigma_{\alpha_2}\cdots\sigma_{\alpha_r} \qquad (w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_r} \text{ reduced}) \tag{53f}$$

for each Weyl group element w. (That σ_w is independent of the choice of reduced decomposition is a consequence of the fact that the σ_α satisfy the braid relations.) If γ is any distinguished (that is, pinning-preserving) automorphism of (G, B, H), then

$$\gamma(\sigma_w) = \sigma_{\gamma(w)}. \tag{53g}$$

The braid relations imply, for any $w \in W$ and simple root α

$$\sigma_w\sigma_\alpha = \begin{cases} \sigma_{ws_\alpha} & \text{length}(ws_\alpha) = \text{length}(w) + 1 \\ \sigma_{ws_\alpha} m_\alpha & \text{length}(ws_\alpha) = \text{length}(w) - 1 \end{cases} \tag{53h}$$

and a similar result for $\sigma_\alpha\sigma_w$ (with m_α on the left).

In exactly the same way, we get a distinguished representative in ${}^\vee G$:

$${}^\vee\sigma_w =_{\text{def}} \sigma_{\alpha_1^\vee}\sigma_{\alpha_2^\vee}\cdots\sigma_{\alpha_r^\vee} \qquad (w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_r} \text{ reduced}). \tag{53i}$$

The main fact we need about these representatives is

Proposition 12.1. *In the setting of* (53)*,*

$$\begin{aligned} \sigma_w\sigma_{w^{-1}} &= (w\rho^\vee - \rho^\vee)(-1) \\ &= e((\rho^\vee - w\rho^\vee)/2) \\ &= \prod_{\substack{\beta \in R^+(G,H) \\ w^{-1}\beta \notin R^+(G,H)}} m_\beta. \end{aligned}$$

The proof is an easy induction on $\ell(w)$. See [2, Lemma 5.4].

Proposition 12.2. *In the setting* (11)*, suppose $w \in W$, $\alpha, \beta \in \Pi$ are simple roots, and $w\alpha = \beta$. Write X_α and X_β for the simple root vectors given by the pinning, and $\sigma_w \in N(H)$ for the Tits representative of w defined in* (53f)*. Then*

$$\sigma_w\sigma_\alpha\sigma_w^{-1} = \sigma_\beta$$

and

$$\mathrm{Ad}(\sigma_w)(X_\alpha) = X_\beta, \quad \mathrm{Ad}(\sigma_w)(X_{-\alpha}) = X_{-\beta}.$$

Proof. Since $\beta = w\alpha$, $s_\beta w = ws_\alpha$. If length$(ws_\alpha) =$ length$(s_\beta w) =$ length$(w) + 1$, then the first case of (53h) implies

$$\sigma_w\sigma_\alpha = \sigma_{ws_\alpha} = \sigma_{s_\beta w} = \sigma_\beta\sigma_w.$$

If the lengths are decreasing, we see

$$\begin{aligned}\sigma_w\sigma_\alpha &= \sigma_{ws_\alpha}m_\alpha \\ &= \sigma_{s_\beta w}m_\alpha \\ &= m_\beta\sigma_\beta\sigma_w m_\alpha \\ &= m_\beta m_{s_\beta w\alpha}\sigma_\beta\sigma_w \\ &= \sigma_\beta\sigma_w \quad \text{(since } s_\beta w\alpha = -\beta).\end{aligned}$$

For the second statement we observe that $\mathrm{Ad}(\sigma_w)(X_\alpha)$ is some multiple of X_β. The Tits group preserves the $\mathbb{Z}$-form of $\mathfrak{g}$ generated by the various $X_{\pm\alpha}$, so this scalar is ± 1; we need to show it is 1. We compute

$$\begin{aligned}\sigma_w\sigma_\alpha\sigma_w^{-1} &= \sigma_w(\exp\frac{\pi}{2}(X_\alpha - X_{-\alpha}))\sigma_w^{-1} \\ &= \exp(\frac{\pi}{2}\,\mathrm{Ad}(\sigma_w)(X_\alpha - X_{-\alpha})).\end{aligned}$$

On the other hand, by what we just proved this equals

$$\sigma_\beta = \exp(\frac{\pi}{2}(X_\beta - X_{-\beta})).$$

Setting these equal gives the two equalities in the second statement. □

Corollary 12.3. *In the setting* (11)*, suppose $w \in W$, $\alpha, \beta \in \Pi$ are simple roots, and $w\alpha = -\beta$. Write X_α and X_β for the simple root vectors given by the pinning, and $\sigma_w \in N(H)$ for the Tits representative of w defined in* (53f)*. Then*

$$\sigma_w\sigma_\alpha\sigma_w^{-1} = \sigma_\beta$$

and

$$\mathrm{Ad}(\sigma_w)(X_\alpha) = -X_{-\beta}, \qquad \mathrm{Ad}(\sigma_w)(X_{-\alpha}) = -X_\beta.$$

Proof. Let $w' = ws_\alpha$. The first assertion follows from the previous lemma applied to $\sigma_{w'}$, using the fact that $\sigma_{w'} = \sigma_w\sigma_\alpha$ (since $w' = ws_\alpha$ is a reduced expression). As in the proof of the previous proposition we conclude that

$$\exp(\frac{\pi}{2}(\mathrm{Ad}(\sigma_w)(X_\alpha - X_{-\alpha})) = \exp(\frac{\pi}{2}(X_\beta - X_{-\beta})),$$

and in this case this implies $\mathrm{Ad}(\sigma_w)(X_\alpha) = -X_{-\beta}$ and $\mathrm{Ad}(\sigma_w)(X_{-\alpha}) = -X_\beta$. □

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Table 1 Types of roots, and their associated Cayley transforms

type	terminology of [10]	definition	Cayley transform
`1C+`	complex ascent	α complex, $\theta\alpha > 0$	
`1C-`	complex descent	α complex, $\theta\alpha < 0$	
`1i1`	imaginary noncpt type I ascent	α imaginary, noncpt, type 1	$\gamma^\kappa = \gamma^\alpha$
`1i2f`	imaginary noncpt type II ascent	α imaginary, noncpt, type 2 δ fixes both terms of γ^α	$\gamma^\kappa = \gamma^\alpha = \{\gamma_1^\kappa, \gamma_2^\kappa\}$
`1i2s`	imaginary noncpt type II ascent	α imaginary, noncpt, type 2 δ switches the two terms of γ^α	
`1ic`	cpt imaginary descent	α cpt imaginary	
`1r1f`	real type I descent	α real, parity, type 1 δ fixes both terms of γ_α	$\gamma_\kappa = \gamma_\alpha = \{\gamma_\kappa^1, \gamma_\kappa^2\}$
`1r1s`	real type I descent	α real, parity, type 1 δ switches the two terms of γ_α	
`1r2`	real type II descent	α real, parity, type 2	$\gamma_\kappa = \gamma_\alpha$
`1rn`	real nonparity ascent	α real, non-parity	
`2C+`	two-complex ascent	α, β complex, $\theta\alpha > 0$, $\theta\alpha \neq \beta$	
`2C-`	two-complex descent	α, β complex, $\theta\alpha < 0$, $\theta\alpha \neq \beta$	
`2Ci`	two-semiimaginary ascent	α, β complex, $\theta\alpha = \beta$	$\gamma^\kappa = s_\alpha \times \gamma = s_\beta \times \gamma$
`2Cr`	two-semireal descent	α, β complex, $\theta\alpha = -\beta$	$\gamma_\kappa = s_\alpha \times \gamma = s_\beta \times \gamma$
`2i11`	two-imaginary noncpt type I-I ascent	α, β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ single valued	$\gamma^\kappa = (\gamma^\alpha)^\beta$
`2i12f`	two-imaginary noncpt type I-II ascent	α, β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ double valued, fixed by δ	$\gamma^\kappa = \{\gamma_1^\kappa, \gamma_2^\kappa\} = (\gamma^\alpha)^\beta$
`2i12s`	two-imaginary noncpt type I-II ascent	α, β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ double valued, switched by δ	
`2i22`	two-imaginary noncpt type II-II ascent	α, β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ has 4 values	$\gamma^\kappa = \{\gamma_1^\kappa, \gamma_2^\kappa\} = \{\gamma^{\alpha,\beta}\}^\sigma$
`2r22`	two-real type II-II descent	α, β real, parity, type 2 $(\gamma_\alpha)_\beta$ single valued	$\gamma_\kappa = (\gamma_\alpha)_\beta$
`2r21f`	two-real type II-I descent	α, β real, parity, type 2 $(\gamma_\alpha)_\beta$ double valued, fixed by δ	$\gamma_\kappa = \{\gamma_\kappa^1, \gamma_\kappa^2\} = (\gamma_\alpha)_\beta$
`2r21s`	two-real type II-I descent	α, β real, parity, type 2 $(\gamma_\alpha)_\beta$ double valued, switched by δ	
`2r11`	two-real type I-I descent	α, β real, parity, type 2 $(\gamma_\alpha)_\beta$ has 4 values	$\gamma_\kappa = \{\gamma_\kappa^1, \gamma_\kappa^2\} = \{(\gamma_\alpha)_\beta\}^\sigma$
`2rn`	two-real nonparity ascent	α, β real, nonparity	
`2ic`	two-imaginary cpt descent	α, β cpt imaginary	
`3C+`	three-complex ascent	α, β complex $\theta\alpha > 0$, $\theta\alpha \neq \beta$	
`3C-`	three-complex descent	α, β complex $\theta\alpha < 0$, $\theta\alpha \neq \beta$	
`3Ci`	three-semiimaginary ascent	α, β complex, $\theta\alpha = \beta$	$\gamma^\kappa = (s_\alpha \times \gamma)^\beta \cap (s_\beta \times \gamma)^\alpha$
`3Cr`	three-semireal descent	α, β complex, $\theta\alpha = -\beta$	$\gamma_\kappa = (s_\alpha \times \gamma)_\beta \cap (s_\beta \times \gamma)_\alpha$
`3i`	three imaginary noncpt ascent	α, β noncpt imaginary, type 1	$\gamma^\kappa = s_\alpha \times \gamma^\beta = s_\beta \times \gamma^\alpha$
`3r`	three-real descent	α, β real, parity, type 2	$\gamma_\kappa = s_\alpha \times \gamma_\beta = s_\beta \times \gamma_\alpha$
`3rn`	three-real non-parity ascent	α, β real, nonparity	
`3ic`	three-imaginary cpt descent	α, β noncpt imaginary	

Table 2 Cayley and cross actions on extended parameters:type `1`

	$\lambda\vert_{\mathfrak{t}} = (\gamma-\rho)\vert_{\mathfrak{t}}$	$({}^\vee\delta_0-1)\lambda = (1+{}^\vee\theta_y)\tau$	$\ell\vert_{\mathfrak{a}} = (g-\rho^\vee)\vert_{\mathfrak{a}}$	$(\delta_0-1)\ell = (1+\theta_x)t$	
type	λ_1	τ_1	ℓ_1	t_1	notes
`1C` crx	$s_\alpha\lambda+(\gamma_\alpha-1)\alpha$	$s_\alpha\tau$	$s_\alpha\ell+(g_\alpha-1)\alpha^\vee$	$s_\alpha t$	
`1i1` crx	λ	τ	$\ell+\alpha^\vee$	t	
`1i1` Cay	λ	$\tau-\tau_\alpha\sigma$	$\ell+\frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee$	t	$\alpha=(1+{}^\vee\theta_{y_1})\sigma$
`1i2f` Cay	$\lambda,\lambda+\alpha$	$\tau-\frac{\tau_\alpha}{2}\alpha$	$\ell+\frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee$	t	τ_α even
`1i2s` Cay	$\lambda,\lambda+\alpha$	[none]	$\ell+\frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee$	t	τ_α odd
`1r1f` Cay	$\lambda+\frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha$	τ	$\ell,\ell+\alpha^\vee$	$t-\frac{t_\alpha}{2}\alpha^\vee$	t_α even
`1r1s` Cay	$\lambda+\frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha$	τ	$\ell,\ell+\alpha^\vee$	[none]	t_α odd
`1r2` crx	$\lambda+\alpha$	τ	ℓ	t	
`1r2` Cay	$\lambda+\frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha$	τ	ℓ	$t-t_\alpha s$	$\alpha^\vee=s+\theta_{x_1}s$

Table 3 Cayley and cross actions on extended parameters: type 2

type	$\lambda\vert_{\mathfrak t} = (\gamma-\rho)\vert_{\mathfrak t}$ λ_1	$({}^\vee\delta_0 - 1)\lambda = (1 + {}^\vee\theta_y)\tau$ τ_1	$\ell\vert_{\mathfrak a} = (g-\rho^\vee)\vert_{\mathfrak a}$ ℓ_1	$(\delta_0 - 1)\ell = (1+\theta_x)t$ t_1	notes
`2C` crx	$w_\kappa\lambda + (\gamma_\alpha - 1)\kappa$	$w_\kappa\tau$	$w_\kappa\ell + (g_\alpha - 1)\kappa^\vee$	$w_\kappa t$	
`2Ci` Cay	$s_\alpha\lambda + (\gamma_\alpha - 1)\alpha$	$s_\alpha\tau - (\lambda_\alpha - \gamma_\alpha + 1)\alpha$	$s_\alpha\ell + (g_\alpha - 1)\alpha^\vee$	$s_\alpha t + (\ell_\alpha - g_\alpha + 1)\alpha^\vee$	$\ell_\alpha = \ell_\beta$
`2Cr` Cay	$s_\alpha\lambda + (\gamma_\alpha - 1)\alpha$	$s_\alpha\tau + (\lambda_\alpha - \gamma_\alpha + 1)\alpha$	$s_\alpha\ell + (g_\alpha - 1)\alpha^\vee$	$s_\alpha t - (\ell_\alpha - g_\alpha + 1)\alpha^\vee$	$\lambda_\alpha = \lambda_\beta$
`2i11` crx	λ	τ	$\ell + \kappa^\vee$	t	
`2i11` Cay	λ	$\tau - \tau_\alpha\sigma(\alpha) - \tau_\beta\sigma(\beta)$	$\ell + \frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee + \frac{g_\beta-\ell_\beta-1}{2}\beta^\vee$	t	$\alpha = (1+{}^\vee\theta_{y_1})\sigma(\alpha)$
`2i12` cr1x	λ	τ	$\ell + \alpha^\vee$	$t - s$	$\alpha^\vee - \beta^\vee = s + \theta_x s$
`2i12f` Cay	$\lambda, \lambda+\alpha$	$\tau + \tau_\beta\sigma$ $- \frac{\tau_\alpha+\tau_\beta}{2}\alpha$, $\tau_1 - \sigma$	$\ell + \frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee$ $+ \frac{g_\beta-\ell_\beta-1}{2}\beta^\vee$	t	$\tau_\alpha + \tau_\beta$ even $\alpha - \beta$ $= \sigma + {}^\vee\theta_{y_1}\sigma$
`2i12s` Cay	$\lambda, \lambda+\alpha$	[none]	$\ell + \frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee + \frac{g_\beta-\ell_\beta-1}{2}\beta^\vee$	t	$\tau_\alpha + \tau_\beta$ odd
`2i22` Cay	$\lambda,\ \lambda+\kappa$ **OR** $\lambda+\alpha,\ \lambda+\beta$	$\tau - \frac{\tau_\alpha}{2}\alpha - \frac{\tau_\beta}{2}\beta$ **OR** $\tau - \frac{\tau_\alpha\pm1}{2}\alpha - \frac{\tau_\beta\mp1}{2}\beta$	$\ell + \frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee$ $+ \frac{g_\beta-\ell_\beta-1}{2}\beta^\vee$	t	τ_α, τ_β even **OR** τ_α, τ_β odd
`2r22` crx	$\lambda + \kappa$	τ	ℓ	t	
`2r22` Cay	$\lambda + \frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha$ $+ \frac{\gamma_\beta-\lambda_\beta-1}{2}\beta$	τ	ℓ	$t - t_\alpha s(\alpha^\vee)$ $- t_\beta s(\beta^\vee)$	$\alpha^\vee =$ $(1+\theta_{x_1})s(\alpha^\vee)$
`2r21` cr1x	$\lambda + \alpha$	$\tau - \sigma$	ℓ	t	$\alpha - \beta = \sigma + {}^\vee\theta_y\sigma$
`2r21f` Cay	$\lambda + \frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha$ $+ \frac{\gamma_\beta-\lambda_\beta-1}{2}\beta$	τ	$\ell, \ell+\alpha^\vee$	$t + t_\beta s$ $- \frac{t_\alpha+t_\beta}{2}\alpha^\vee$, $t_1 - s$	$t_\alpha + t_\beta$ even $\alpha^\vee - \beta^\vee$ $= s + \theta_{x_1}s$
`2r21s` Cay	$\lambda + \frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha$ $+ \frac{\gamma_\beta-\lambda_\beta-1}{2}\beta$	τ	$\ell, \ell+\alpha^\vee$	[none]	$t_\alpha + t_\beta$ odd
`2r11` Cay	$\lambda + \frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha$ $+ \frac{\gamma_\beta-\lambda_\beta-1}{2}\beta$	τ	$\ell,\ \ell+\kappa^\vee$ **OR** $\ell+\alpha^\vee,\ \ell+\beta^\vee$	$t - \frac{t_\alpha}{2}\alpha^\vee - \frac{t_\beta}{2}\beta^\vee$ **OR** $t - \frac{t_\alpha\pm1}{2}\alpha^\vee - \frac{t_\beta\mp1}{2}\beta^\vee$	t_α, t_β even **OR** t_α, t_β odd

Table 4 Cayley and cross actions on extended parameters: type 3

type	$\lambda\|_{\mathfrak{t}} = (\gamma - \rho)\|_{\mathfrak{t}}$ λ_1	$({}^\vee\delta_0 - 1)\lambda = (1 + {}^\vee\theta_y)\tau$ τ_1	$\ell\|_{\mathfrak{a}} = (g - \rho^\vee)\|_{\mathfrak{a}}$ ℓ_1	$(\delta_0 - 1)\ell = (1 + \theta_x)t$ t_1	notes
3C crx	$w_\kappa\lambda + (\gamma_\kappa - 2)\kappa$	$w_\kappa\tau$	$w_\kappa\ell + (g_\kappa - 2)\kappa^\vee$	$w_\kappa t$	
3Ci Cay	λ **OR** $\lambda + \kappa$	$\tau - \frac{\tau_\kappa}{2}\kappa$	$\ell + (g_\alpha - 1 - \ell_\alpha)\kappa^\vee$	t	$\gamma_\alpha - 1 - \lambda_\alpha$ even **OR** $\gamma_\alpha - 1 - \lambda_\alpha$ odd
3Cr Cay	$\lambda + (\gamma_\alpha - 1 - \lambda_\alpha)\kappa$	τ	ℓ **OR** $\ell + \kappa^\vee$	$t - \frac{t_\kappa}{2}\kappa^\vee$	$g_\alpha - 1 - \ell_\alpha$ even **OR** $g_\alpha - 1 - \ell_\alpha$ odd
3i Cay	λ	$\tau - \tau_\kappa\alpha$	$\ell + \left(g_\alpha - 1 - \frac{\ell_\kappa}{2}\right)\kappa^\vee$	t	
3r Cay	$\lambda + \left(\gamma_\alpha - 1 - \frac{\lambda_\kappa}{2}\right)\kappa$	τ	ℓ	$t - t_\kappa\alpha^\vee$	

Table 5 Action of Hecke operators

κ-type(E)	$T_\kappa(E)$	κ-type(E)	$T_\kappa(E)$
`1C+`	$w_\kappa \times E$	`1C-`	$(q-1)E + q(w_\kappa \times E)$
`1i1`	$w_\kappa \times E + E_\kappa$	`1r2`	$(q-1)E - w_\kappa \times E + (q-1)E_\kappa$
`1i2f`	$E + E_\kappa^1 + E_\kappa^2$	`1r1f`	$(q-2)E + (q-1)(E_\kappa^1 + E_\kappa^2)$
`1i2s`	$-E$	`1r1s`	qE
`1ic`	qE	`1rn`	$-E$
`2C+`	$w_\kappa \times E$	`2C-`	$(q^2-1)E + q^2(w_\kappa \times E)$
`2Ci`	$qE \pm (q+1)E_\kappa$ (see Section 9)	`2Cr`	$(q^2-q-1)E \pm (q^2-q)E_\kappa$ (see Section 9)
`2i22`	$E + E_\kappa^1 + E_\kappa^2$	`2r11`	$(q^2-2)E + (q^2-1)(E_\kappa^1 + E_\kappa^2)$
`2i11`	$w_\kappa \times E + E_\kappa$	`2r22`	$(q^2-1)E - w_\kappa \times E + (q^2-1)E_\kappa$
`2i12f`	$w_\kappa \times E \pm E_\kappa^1 \pm E_\kappa^2$ (see Section 8)	`2r21f`	$(q^2-2)E + (q^2-1)(\pm E_\kappa^1 \pm E_\kappa^2)$ (see Section 8)
`2i12s`	$-E$	`2r21s`	q^2E
`2ic`	q^2E	`2rn`	$-E$
`3C+`	$w_\kappa \times E$	`3C-`	$(q^3-1)E + q^3(w_\kappa \times E)$
`3Ci`	$qE + (q+1)E_\kappa$	`3Cr`	$(q^3-q-1)E + (q^3-q)E_\kappa$
`3i`	$qE + (q+1)E_\kappa$	`3r`	$(q^3-q-1)E + (q^3-q)E_\kappa$
`3ic`	q^3E	`3rn`	$-E$