Affine Weyl group and coherent families

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1 Weyl groups

Let $G$ be a complex reductive group, with dual group $G^\vee$, Weyl group $W$, character lattice $X^*$, and root lattice $R$
The affine Weyl group is $W^{aff} := W \ltimes R$
The extended affine Weyl group is $W^{ext} := W \ltimes X^*$
They act on $X^*$, $\mathfrak{h}_R^* := X^* \otimes_{\mathbb{Z}} \mathbb{R}$, and $\mathfrak{h}^* := X^* \otimes_{\mathbb{Z}} \mathbb{C}$

$W^{aff}/W^{ext} \approx X^*/R \approx$ algebraic characters of $Z(G)$
$\mathfrak{h}_R^*/W^{aff}$ is the fundamental alcove
$\mathfrak{h}^*/W^{ext} \leftrightarrow$ conjugacy classes of semisimple elements in $G^\vee$
$\mathfrak{h}^*/W \leftrightarrow$ characters of $Z(g)$ (Harish-Chandra isomorphism)

2 Coherent family

Let $\Xi$ be an $X^*$-coset in $\mathfrak{h}^*/X^*$.
A coherent family for $G_\mathbb{R}$ based on $\Xi$ is a map
$\Theta : \Xi \to \{\text{virtual representations of } G_\mathbb{R}\}$ such that

1. $\Theta (\xi)$ has infinitesimal character $\xi$, for all $\xi$ in $\Xi$
2. for any finite dimensional representation $F$ of $G$ we have
   $F \otimes \Theta (\xi) = \sum_\mu m_F(\mu) \Theta (\xi + \mu)$
   where $m_F(\mu)$ is the multiplicity of the weight $\mu$ in $F$.

The set of all such $\Theta$ forms a $\mathbb{Z}$-module $CF(\Xi)$.
[More generally we can consider coherent families with values in other categories of representations]

Problem 1 Define a $W^{aff}$ representation on $CF(\Xi)$
3 \( W \) action

Given \( w \in W \) and \( \Theta \) in \( CF(\Xi) \), define
\[
(w\Theta)(\xi) := \Theta(w^{-1}\xi) \quad \text{for} \quad \xi \in \Xi
\]
Then \( w\Theta \) is a coherent family based on \( w\Xi \)
This defines a \( W \) representation on the sum of the various \( CF(w\Xi) \) as \( w \) ranges over \( W \)

However, consider the following subgroups of \( W \):
\[
W_{[\Xi]} := \{ w \in W : w(\Xi) = \Xi \}
\]
\[
W_{\Xi} := \{ w \in W : w(\xi) - \xi \in R \text{ for some (hence all) } \xi \in \Xi \}
\]
Then these act naturally on \( CF(\Xi) \) itself.
\( [W_{\Xi} \text{ is a parabolic subgroup of } W_{aff}, \text{ though not of } W] \)

Let \( s = s(\Xi) \) be the semisimple element in \( H^\vee \subset G^\vee \)
corresponding to the natural map \( h^*/X^* \to h^*/W_{ext} \).
Its centralizer \( G^\vee_s \) is reductive (perhaps disconnected).
Then \( W_{[\Xi]} \) is the Weyl group \( W(G^\vee_s, H^\vee) \) and
we have \( W_{[\Xi]}/W_{\Xi} \approx G^\vee_s/\{\text{identity component}\} \)

4 Associated variety and cycle

Suppose \( \xi_0 \in \Xi \) is regular. The map \( \Theta \to \Theta(\xi_0) \) gives a bijection:
\( CF(\Xi) \to \{ \text{Virtual representations with infinitesimal character } \xi_0 \} \)
Given \( \Theta \) in \( CF(\Xi) \), write \( \Theta(\xi_0) = \sum m_i X_i \), where \( X_i \) are irreducible
The associated variety of \( X_i \) is a union of closures of nilpotent \( K \) orbits
Let \( O_1, \ldots, O_n \) be the maximal such orbits (obtained as \( i \) varies).
This collection is independent of the regular \( \xi_0 \in \Xi \)
Write \( O_j \approx K/K_j \) where \( K_j \) is the stabilizer of some point in \( O_j \).

The associated cycle of \( \Theta \) gives for each \( \xi \) in \( \Xi \),
a virtual representation \( \tau_j(\xi) \) of each \( K_j \).
We define \( \tau_j(\xi) = \sum m_i \tau_{ij}(\xi), \) then we have
\[
\tau_j(\xi) \otimes F = \sum_{\mu} m_F(\mu) \tau_j(\xi + \mu)
\]
We want to compute \( \tau_j(\xi) \); the following result should be useful:

**Proposition 2** If \( \phi : X^* \to \mathbb{C} \) is a function satisfying
\[
\dim(F) \phi(\lambda) = \sum_{\mu} m_F(\mu) \phi(\lambda + \mu)
\]
Then \( \phi \) is a harmonic polynomial in \( S(h) \).