Software Project: Weakly Fair $A_{\mathfrak{q}}(\lambda)$

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This is a project recommended by David Vogan at the Atlas workshop, July 2006.

Suppose $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$, and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{l}$. Suppose $\lambda \in \mathfrak{h}^*$ defines a one-dimensional representation of $\mathfrak{l}$, i.e. $\langle \lambda, \alpha^\vee \rangle = 0$ for all roots of $\mathfrak{h}$ in $\mathfrak{l}$. Fix $\Delta^+ = \Delta^+(\mathfrak{h}, \mathfrak{g})$ containing $\Delta(\mathfrak{h}, \mathfrak{u})$, and let $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ and $\rho(\mathfrak{u}) = \frac{1}{2}\sum_{\alpha \in \Delta(\mathfrak{h},\mathfrak{u})} \alpha$ as usual. We normalize cohomological induction so $A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda + \rho$.

We say λ is *good* if

$$\langle \lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Delta^+ \tag{1}$$

and is *weakly fair* if

$$\langle \lambda + \rho(\mathfrak{u}), \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta^+ \tag{2}$$

If λ is good then $A_{\mathfrak{q}}(\lambda)$ is non-zero, irreducible, unitary, and has regular integral infinitesimal character. These representations are quite well understood. If λ is weakly fair then $A_{\mathfrak{q}}(\lambda)$ is non-zero, unitary, but not necessarily irreducible. It is of interested to compute this sum: some of these are interesting unitary representations.

That is write

$$A_{\mathfrak{q}}(\lambda) = \sum_i a_i \pi_i \tag{3}$$

where each π_i is irreducible, and $a_i \in \mathbb{N}$.

Problem: Use the atlas software to compute (3).

Here is a sketch of what is involved.

Fix $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and weakly fair λ.

This is all going on in the block of the trivial representation, so label the parameters in this block $0, \dots, n$. Write $\pi(i)$ for the irreducible representation with parameter i, and $I(i)$ for the standard representation.

Recall $A_{\mathfrak{q}}(0)$ has infinitesimal character ρ and is unitary. It occurs in the output of the `blocku` command. Find i (in the output of `blockd` or `blocku`) so that $A_{\mathfrak{q}}(0) = \pi(i)$.

Now choose $w \in W$ so that

$$\langle w(\lambda + \rho), \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta^+. \tag{4}$$

Next compute

$$w^{-1} \cdot \pi(i) = \sum_{j \in S} c_j \pi(j) \tag{5}$$

where $S \subset \{0, \dots, n\}$. Here $w^{-1} \cdot \pi(i)$ is the coherent continuation action. The `wgraph` command gives the information needed to compute this action. This is carried out by the helper application `coherentContinuation` at `www.liegroups.org/software/helpers` (at some point this will be built into `atlas` itself).

Now list the simple roots $\beta_1, \dots, \beta_r$ such that

$$\langle w(\lambda + \rho), \beta_k^\vee \rangle = 0. \tag{6}$$

Now discard those $\pi(j)'s$ for which some β_k is in the τ-invariant of $\pi(j)$. That is write $S' \subset S$ for the set of $j \in S$ for which

$$\text{for all } 1 \leq k \leq r, \beta_k \text{ is not in the } \tau \text{ invariant of } \pi(j) \tag{7}$$

This is available from the output of the `block` command, see the Appendix. We discard $\pi(j)$ if:

$$\text{some } \beta_k \text{ is of type } ic, r1, r2, \text{ or } C- \text{ for } \pi(j). \tag{8}$$

Then we have

$$A_{\mathfrak{q}}(\lambda) = \sum_{j \in S'} c_j \pi'(j) \tag{9}$$

where

$$\pi'(j) = \psi^{\rho}_{w(\lambda+\rho)}(\pi(j)) \tag{10}$$

the translation of $\pi(j)$ at infinitesimal character ρ to infinitesimal character $\lambda + \rho$. Each $\pi'(j)$ is irreducible, and (by the assumption on the τ-invariant) non-zero.

1 Appendix

We take the opportunity to summarize in one place information about types of roots. Fix a block with parameters $0 \le i \le n$. Recall the cross action is defined on parameters; we write $w \times i = j$.

For each parameter i each simple root is listed by type in the output of `block`:

- i1: α is imaginary, noncompact, and type I, meaning the following equivalent conditions hold:
  - (a) $s_\alpha \notin W(G(\mathbb{R}), H(\mathbb{R}))$
  - (b) the Cayley transform c_α is single-valued
  - (c) $s_\alpha \times i \neq i$
  - (d) this is an $SL(2,\mathbb{R})$-situation
  - (e) $c_\alpha(\alpha)$ is of type r1
- i2: α is imaginary, noncompact, and type II, meaning the following equivalent conditions hold:
  - (a) $s_\alpha \in W(G(\mathbb{R}), H(\mathbb{R}))$
  - (b) the Cayley transform c_α is double-valued
  - (c) $s_\alpha \times i = i$
  - (d) this is a $PGL(2,\mathbb{R})$ situation
  - (e) $c_\alpha(\alpha)$ is of type r2
- r1: α is real, satisfies the parity condition, and is type 1, meaing the following equivalent conditions hold:
  - (a) $\alpha(h) \neq -1$ for any $h \in H(\mathbb{R})$,
  - (b) the Cayley transform c^α is double valued
  - (c) $s_\alpha \times i = i$
  - (d) this is an $SL(2,\mathbb{R})$-situation
  - (e) $c^\alpha(\alpha)$ is of type i1
- r2: α is real, satisfies the parity condition, and is of type 2, meaing the following equivalent conditions hold:

(a) $\alpha(h) = 1$ for some $h \in H(\mathbb{R})$,

(b) the Cayley transform c^α is single valued

(c) $s_\alpha \times i \neq i$

(d) this is a $PGL(2, \mathbb{R})$-situation

(e) $c^\alpha(\alpha)$ is of type i2

rn: α is real, and does not satisfy the parity condition.

ic: α is imaginary and compact,

C+: α is complex, and $\theta(\alpha) > 0$

C-: α is negative, and $\theta(\alpha) < 0$

The roots in the τ-invariant of the irreducible representation $\pi(i)$ with parameter i are determined as follows. If α is simple then:

1. α imaginary: $\alpha \in \tau(\pi(i)) \Leftrightarrow \alpha$ is compact
2. α real: $\alpha \in \tau(\pi(i)) \Leftrightarrow \alpha$ satisfies the parity condition
3. α complex: $\alpha \in \tau(\pi(i)) \Leftrightarrow \theta(\alpha) < 0$

In the notation of atlas this means

$$\begin{aligned} \tau(\pi(i)): &\quad ic, r1, r2, C- \\ \text{not in } \tau(\pi(i)): &\quad i1, i2, rn, C+ \end{aligned} \tag{11}$$