# Petite and Relevant $K$-types for exceptional groups

**Alessandra Pantano**
joint work with Dan Barbasch

July 2005

## Introduction

Let $G$ be a real split group, with Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $K$ be the maximal compact subgroup of $G$ and let $M$ be the centralizer in $K$ of a maximal abelian subspace of $\mathfrak{p}$.
If $\delta$ is a representation of $M$, we denote by $W_\delta^0$ the Weyl group of good coroots for $\delta$.

For every petite $K$-type $\mu$ containing $\delta$, there is a representation of $W_\delta^0$ on the the space $\mathrm{Hom}_M(E_\mu, V^\delta)$. We ask whether all the relevant $W_\delta^0$-types can be realized this way.

**Question:** Given any *relevant* representation $\tau$ of $W_\delta^0$, is there a *petite* $K$-type $\mu$ such that $\mathrm{Hom}_M(E_\mu, V^\delta) = \tau$?

# Motivation

Let $\mathbb{H}_\delta$ be the p-adic split group associated to the root system of the good co-roots. Suppose that

- Every relevant $W_0^\delta$ type $\tau$ appears in $\mathrm{Hom}_M(E_\mu, V^\delta)$, for some petite $K$-type $\mu_\tau$
- Every relevant $W_0^\delta$ type $\tau$, the intertwining operator on $\mu_\tau$ matches the p-adic operator on $\tau$.

Then we conclude that the (possibly non-spherical) Langlands quotient $\bar{X}(\delta, \nu)$ for $G$ (real) is unitary *only* if the spherical Langlands quotient $\bar{X}(\delta, \nu)$ for $\mathbb{H}_\delta$ (p-adic) is unitary.

**This is a non-unitarity certificate for $\bar{X}_P(\delta, \nu)$.**

# Non-unitarity certificates for Langlands quotients

the real Langlands quotient $\bar{X}(\delta \otimes \nu)$ is unitary

$\Updownarrow$

$R_\mu(\omega, \nu)$ is positive semidefinite, for all $K$-types $\mu$

$\Downarrow$

$R_\mu(\omega, \nu)$ is positive semidefinite, for all *relevant* $K$-types $\mu$

$\Updownarrow$

$R_\tau(\omega, \nu)$ is positive semidefinite, for all *relevant* $W$-types $\tau$

$\Updownarrow$

the p-adic Langlands quotient $\bar{X}(\nu)$ is unitary

# A remark

The previous argument gives a way to compare the unitarity of a *(possibly) non-spherical* Langlands quotient of the *real group* G with the unitarity of a *spherical* Langlands quotient of the *p-adic group* $\mathbb{H}_\delta$.

If the root system of the good co-roots $\Delta_\delta$ is of classical type, we can replace $\mathbb{H}_\delta$ with the *real* split group $\mathbb{G}_\delta$ associated to $\Delta_\delta$. Then the comparison remains in the category of *real* split groups:

$$\boxed{\text{the real Langlands quotient } \bar{X}(\delta \otimes \nu) \text{ is unitary for } G}$$

$$\Downarrow$$

$$\boxed{\text{the real Langlands quotient } \bar{X}(triv. \otimes \nu) \text{ is unitary for } \mathbb{G}_\delta}$$

# The Problem

*Now that the motivation is understood, we describe the problem addressed in this talk...:*

Let $G$ be the double cover of a real split group of type $E_6$, $E_7$, $E_8$ or $F_4$. Given *any irreducible representation* $\delta$ *of* $M$, and *any petite* $K$*-type* $\mu$ containing $\delta$, compute the representation of $W_\delta^0$ on the space $\mathrm{Hom}_M(E_\mu, V^\delta)$.

This is a complicated problem. We divide it in several steps.

*step 1* Identify **fine** and **petite** $K$-types

$\Downarrow$

*step 2* Understand $\bigotimes$ of repr.s of M, and find restriction of $K$-types to $M$

$\Downarrow$

*step 3* Find the representation of $W_\delta^0$ on the isotypic component of $\delta$, for $\delta$ **trivial** or **genuine**

$\Downarrow$

*step 4* Complete the work (for other $\delta$'s)

## ***step 1:*** **Identify fine and petite $K$-types**

*We work with the double cover...*

- Classify $\tilde{K}$-types (highest weight or fundamental weights) and find a formula to compute the level of a $\tilde{K}$-type
- **Fine** $\tilde{K}$-types have level 0, $\frac{1}{2}$ or 1
- **Petite** $\tilde{K}$-types have level 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2 or 3

## *step 2:* Understand $\otimes$ of repr.s of $\tilde{M}$, and find the restriction of $\tilde{K}$-types to $\tilde{M}$

- Restrict fine $\tilde{K}$-types to $\tilde{M}$ ($\rightarrow$ orbits of a single $\tilde{M}$-type). Each $\tilde{M}$-type $\delta$ appears in at least one fine $\tilde{K}$-type $\mu_\delta$
- To find $\delta_1 \otimes \delta_2$, look at the tensor product $\mu_{\delta_1} \otimes \mu_{\delta_2}$ and restrict the summands to $M$
- To find $Res_{\tilde{M}} \mu$, use an inductive algorithm:
  - Embed $\mu$ in a tensor product of fine $\tilde{K}$-types
  - Decompose the tensor products (using *LiE*)
  - Restrict the summands to $M$ and guess how the various repr.s of $M$ distribute among the composition factors...

♠ *Problem: fine $K$-types don't generate the Grothendieck group!*

## ***step 3:*** Find the repr. of $W_\delta^0$ on the $\delta$-isotypic, for $\delta$ trivial or genuine

*We work simultaneously with spherical and genuine $K$-types. Induction, restriction and tensor product of Weyl group representations are computed using GAP.*

This is the algorithm used for $E_6$ and $E_8$ (the easiest cases):

1. Any $\delta$ is included in *one* fine $K$-type $\mu_\delta$. As a $W$-representation:

$$(\mu_\delta \otimes \mu_\delta^\star)^M = Ind_{W_\delta^0 = W^\delta}^W(trivial).$$

You get the action of $W$ on $\mu^M$, for all $K$-types $\mu$ in $(\mu_\delta \otimes \mu_\delta^\star)$

$\Downarrow$

$$\Downarrow$$

2. There is *one* genuine $M$-type $\delta_g$. For $\Theta$ genuine, $\Theta\mid_M = a\delta_g$, so

$$\mathrm{Hom}_M(\Theta,\delta_g) = \mathrm{Hom}_M(\Theta,\mu_{\delta_g}) = (\Theta\otimes\mu_{\delta_g}^\star)^M \longleftarrow \textit{known, by (1)}$$

You get the action of $W_0^{\delta_g} = W$ on $V_\Theta(\delta_g)$, for some $\Theta$ genuine

$$\Updownarrow$$

3. If $\Theta_1$, $\Theta_2$ are genuine

$$(\Theta_1\otimes\Theta_2^\star)^M = \mathrm{Hom}_M(\Theta_1,\Theta_2) = \overbrace{\mathrm{Hom}_M(\Theta_1,\delta_g)\otimes\mathrm{Hom}_M(\delta_g,\Theta_2)}^{\textit{known, by (2)}}$$

You get the action of $W$ on $\mu^M$, for all $K$-types $\mu$ in $(\Theta_1\otimes\Theta_2^\star)$

♠ *The algorithm is bit harder for $E_7$, and a lot harder for $F_4$.*

## Find the repr. of $W^0_\delta$ on the $\delta$-isotypic, for $\delta$ **trivial** or **genuine**

*Modifying the algorithm for* $E_7 \ldots$

- For $E_7$, $R_\delta$ can have order two. In this case $\delta$ is contained in two fine $K$-types, and

$$\boxed{Ind^W_{W^0_\delta}(trivial) = (\mu^1_\delta \otimes (\mu^1_\delta)^\star)^M + (\mu^1_\delta \otimes (\mu^2_\delta)^\star)^M}$$

- $E_7$ has two genuine $M$-types, both with $W^0_\delta = W$. The relation

$$\boxed{\mathrm{Hom}_M(\Theta_1, \Theta_2) = \mathrm{Hom}_M(\Theta_1, \delta_g) \otimes \mathrm{Hom}_M(\delta_g, \Theta_2)}$$

works *only* if $\Theta_1 \mid_M = a\delta_g$ and $\Theta_2 \mid_M = b\delta_g$.

## Find the repr. of $W_\delta^0$ on the $\delta$-isotypic, for $\delta$ **trivial** or **genuine**

The case of $F_4$ is by far the hardest:

- if $\mu$ is genuine, $\mu\,|_M = a\delta_2 + b\delta_6$ (not isotypic ...)
- the genuine $M$-type $\delta_6$ has $W_{\delta_6}^0 \neq W$.

The other genuine $M$-type has $W_{\delta_2}^0 = W$, and the isomorphism

$$\boxed{\mathrm{Hom}_M(\Theta_1, \Theta_2) = \mathrm{Hom}_M(\Theta_1, \delta_2) \otimes \mathrm{Hom}_M(\delta_2, \Theta_2)}$$

works only if $\Theta_1\,|_M = a\delta_2$ and $\Theta_2\,|_M = b\delta_2 + c\delta_6$.
When $\Theta_1\,|_M = a\delta_6$ and $\Theta_2\,|_M = b\delta_2 + c\delta_6$, we can use:

$$\boxed{\mathrm{Hom}_M(\Theta_1, \Theta_2) = \mathrm{Ind}_{W(B_4)}^{W} \left[\mathrm{Hom}_M(\Theta_1, \delta_6) \otimes \mathrm{Hom}_M(\delta_6, \Theta_2)\right] .}$$

*Sometimes it is useful to look at the inclusion of $F_4$ into $E_6$.*

## ***step 4:*** Find the repr. of $W_\delta^0$ on the $\delta$-isotypic, for $\delta$ non-trivial and non-genuine

*We only discuss the easiest case $E_6$. A similar, but more complicated argument works for other groups.*

Suppose that $\delta_g$ is genuine for $M$, $\Theta$ is genuine for $K$ and $\Theta\,|_M$ contains $\delta_g$. Then $(\mu_{\delta_g} \otimes \Theta)\,|_M$ contains $\delta$, and

$$\mathrm{Hom}_M(\delta, \mu_{\delta_g} \otimes \Theta) = \mathrm{Res}_{W_\delta^0}^{W_{\delta_g}^0 = W} \underbrace{\mathrm{Hom}_M(\delta_g, \Theta)}_{known,\, by\, step\, 3}.$$

We can use this isomorphism to compute the repr. of $W_\delta^0$ on the isotypic component of $\delta$ in the composition factors of $\mu_{\delta_g} \otimes \Theta$.

♠ *This gets very tricky, especially for $F_4$ (too much ambiguity. . . )*