A parametrization of $\widehat{K}$ (after Vogan)

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Let G be a real reductive group in Harish-Chandra's class. It may be instructive and useful to weaken that hypothesis, but we content ourselves with it here.

Let K be the maximal compact subgroup of G. The point of these notes is to recall a parametrization of $\widehat{K}$ (i.e. equivalence classes of irreducible representations of K) due to David Vogan. Note that even if G is algebraic, the description of $\widehat{K}$ is not covered by Adams' notes on parameters: the group K need not belong to the class Adams considers (even though the group G does).

For orientation one should consult the notes on $\widehat{K}$ compiled last year by David Vogan during the AIM workshop. (These are on the website.) Those notes provide provide a completely different perspective, essentially that of Cartan-Weyl, and parametrize $\widehat{K}$ in terms of irreducible representations of a Cartan subgroup. By contrast, these notes intricately use the fact that our K is the maximal compact subgroup of G.

Theorem 1 *Let $\widehat{G}^{\mathrm{temp},\circ}$ denote the set of irreducible tempered representations with real infinitesimal character. Then the map*

$$\widehat{G}^{\mathrm{temp},\circ} \longrightarrow \widehat{K}$$

obtained by taking lowest K-types is a well-defined bijection. More precisely, if $\overline{X} \in \widehat{G}^{\mathrm{temp},\circ}$, then

1. *$\overline{X}$ has a unique lowest K-type;*
2. *Two irreducible tempered representations with real infinitesimal character whose lowest K type coincide are necessarily isomorphic; and*
3. *Each K type $\mu \in \widehat{K}$ arises as the lowest K-type of an element of $\widehat{G}^{\mathrm{temp},\circ}$*

Thus $\widehat{K}$ is parametrized by $\widehat{G}^{\mathrm{temp},\circ}$. A parametrization of this latter set in terms of (more or less) combinatorial data is given in Proposition 6. Putting them together we get the parametrization we seek (Corollary 7).

The perspective offered by Theorem 1 has a number of wonderful advantages. For instance it seems plausible that in our software we may not need to compute $\widehat{T}$ explicitly for T a large Cartan subgroup of K. (Even determining the component group of T, not to mention its structure as a finite group, is complicated.) As we explain below, using the parametrization of $\widehat{K}$ in terms of $\widehat{G}^{\mathrm{temp},\circ}$ makes computing lowest K-types essentially trivial.

Not surprisingly, the proofs of each of the statements in Theorem 1 are closely tied to parametrizing $\widehat{G}^{\mathrm{temp},\circ}$ and in fact all irreducible admissible representations of G. It is difficult to find precise references which extract exactly the statement above, but here is a guide. Recall that every irreducible tempered representation occurs in a module induced from a cuspidal parabolic MAN and a discrete series representation $\delta \otimes \mathbb{1} \otimes \mathbb{1}$. (The point is that in order to be tempered the exponential e^{ν} on A must have ν imaginary; in order to have real infinitesimal character ν must be real. So $\nu = 0$ and $e^{\nu} = \mathbb{1}$.) Sorting out the irreducible constituents is handled by the R-group. This was done for connected linear G by Knapp-Stein; Vogan and others extended it to the present context (where G need be neither connected nor linear). We are going to recall this momentarily. For now, we remark that Theorem 1(1)–(2) are a consequence of the classification of $\widehat{G}^{\mathrm{temp},\circ}$ and [Vunit, Theorem 3.40]. As for Theorem 1(3) below, the best place to start to look is around Theorem 6.5.12 in [Vgr].

Example 2 For orientation, we include the example of $G = \mathrm{SL}(2,\mathbb{R})$. Of course $K = \mathrm{SO}(2)$ and $\widehat{K}$ is parametrized by $\mathbb{Z}$. In the obvious notation, write $\mu_n \in \widehat{K}$. The relevant observation is that for $|n| \geq 2$, μ_n is the LKT of a (tempered) discrete series. Meanwhile the two representations $\mu_{\pm 1}$ arise as LKTs of the two (tempered) irreducible limits of discrete series. (We may also realize the pair $\mu_{\pm 1}$ as the LKTs of the reducible nonspherical (tempered) principal series with infinitesimal character zero; this is where the R-group shenanigans first appear.) Finally, the trivial representation μ_0 is the LKT of the (tempered) irreducible spherical principal series with infinitesimal character zero. In particular, we see that by passage to lowest K-types, we obtain a bijection from the set of irreducible tempered representation of G with real infinitesimal character to $\widehat{K}$.

It's also a good idea to keep $\mathrm{GL}(2,\mathbb{R})$ in mind; here K is the disconnected orthogonal group $\mathrm{O}(2)$. This time $\widehat{K}$ is parametrized by strictly positive integers, together with trivial and sgn representations which we denote $\mu_0^{\pm}$. For $n > 0$, the K type μ_n arises as the LKT of relative discrete series (or in the case of μ_1, relative limits of discrete series). To account for $\mu_0^{\pm}$, note that there are four tempered principal series with infinitesimal character zero corresponding to the four characters of M. Two of these principal series are isomorphic and isomorphic to a relative limit of discrete series; so we have already accounted for them. The other two are distinct; one has LKT μ_0^{+}, the other μ_0^{-}.

Finally we consider $\mathrm{U}(1,1)$ to illustrate how the parametrization behaves in the rank one case even when we restrict to connected groups. In this case $K = \mathrm{U}(1) \times \mathrm{U}(1)$ and a K-type is thus a pair of integers $(a,b) \in \mathbb{Z}^2$. Suppose $\lambda = (\lambda_1, \lambda_2)$ is the Harish-Chandra parameter of a discrete series or a limit of discrete series. So $\lambda_i \in \frac{1}{2} + \mathbb{Z}$, and there are two cases:

$\lambda_1 \geq \lambda_2$. The LKT of the corresponding discrete series is

$$\lambda + \rho = \left(\lambda_1 + \frac{1}{2}, \lambda_2 - \frac{1}{2}\right).$$

Hence we obtain all K types of the form $(a,b) \in \mathbb{Z}^2$ with $a > b$.

$\lambda_1 \leq \lambda_2$. The LKT of the corresponding discrete series is

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Hence we obtain all K types of the form $(a, b) \in \mathbb{Z}^2$ with $a < b$.

Thus the discrete series and limits parameterize all K types of the form (a, b) with $a \neq b$. We are missing those with $a = b$, but it is obvious where to find them. The irreducible spherical principal series with infinitesimal character zero give us the trivial K type $(0, 0)$, and the others are obtained by tensoring with powers of the determinant. $\square$

Now we turn to parametrizing $\widehat{G}^{\mathrm{temp},\circ}$. To begin, we need to discuss how to parametrize *all* irreducible admissible representations of G. We are going to trot out pseudocharacters; these are different from the parameters in Adams' notes, but translating between the two is not so serious. (It would be good to work this out explicitly, at least in the case of tempered representations, as an example in Adams' notes.) The main point is that all the conditions we impose on our parameters also translate nicely into Adams' parameters.

Write θ for the Cartan involution of G. Let $\mathfrak{h}_\circ = \mathfrak{t}_\circ \oplus \mathfrak{a}_\circ$ denote a θ-stable Cartan in $\mathfrak{g}_\circ$, the Lie algebra of G. As usual drop $\circ$ subscripts to denote complexifications. Let H denote the centralizer of $\mathfrak{h}_\circ$ in G. The decomposition $\mathfrak{h}_\circ = \mathfrak{t}_\circ \oplus \mathfrak{a}_\circ$ implies that $H = TA$ where $T = H \cap K$ and $A = \exp(\mathfrak{a}_\circ)$ is a vector group. A *regular pseudocharacter* of H is a pair

$$\gamma = (\Gamma, \bar{\gamma})$$

subject to the following conditions:

(R1) Γ is an irreducible representation of H and $\bar{\gamma} \in \mathfrak{h}^*$;

(R2) Suppose α is an imaginary root of $\mathfrak{h}$ in $\mathfrak{g}$. Then $\langle \bar{\gamma}, \alpha \rangle$ is real and nonzero, and hence $\bar{\gamma}$ defines a system of positive roots Ψ (making $\bar{\gamma}$ dominant);

(R3) If we write $\rho(\Psi)$ for the half-sum of the elements of Ψ and $\rho_c(\Psi)$ for the half-sum of the compact ones, then

$$d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi). \tag{3}$$

Write $\mathcal{P}_{\mathrm{reg}}(H)$ for the set of regular pseudocharacters attached to H.

To each $\gamma = (\Gamma, \bar{\gamma}) \in \mathcal{P}_{\mathrm{reg}}(H)$, we may build a standard module $X(\gamma)$ as follows. Let L denote the centralizer in G of A. Conditions (R2) and (R3) imply that $\bar{\gamma}$ is the Harish-Chandra parameter of a discrete series for L. Since L may be disconnected, the Harish-Chandra parameter $\bar{\gamma}$ need not determine a single discrete series. We may, however, specify such a discrete series, say X^L, by requiring that its lowest $L \cap K$ type have highest weight Γ. Next choose a real parabolic subgroup LN so

$$\text{the real part of } \bar{\gamma} \text{ restricted to } \mathfrak{a}_\circ \text{ is negative on the roots of } \mathfrak{a} \text{ in } \mathfrak{n}. \tag{4}$$

Define

$$X(\gamma) = \mathrm{ind}_{LN}^{G}(X^L \otimes 1\!\!1).$$

Then $X(\gamma)$ has infinitesimal character $\bar{\gamma}$ and the condition in (4) guarantees that it has a unique submodule.

The standard modules $X(\gamma)$ for $\gamma \in \mathcal{P}_{\mathrm{reg}}$ are enough for some purposes, but not enough for a classification. For instance, for $\mathrm{SL}(2,\mathbb{R})$, the only way to get the two limits of discrete series is as the two constituents of the (reducible) nonspherical principal series with infinitesimal character zero. Thus if we are interested in a map from our standard modules to irreducibles, it must be multivalued. To remedy this, we must enlarge the class of standard modules (by considering "limit" pseudocharacters); in the SL(2) case, this amounts to including the two limits of discrete series as standard modules. To make a bijection between standard and irreducibles, we must then throw out some standard modules our (by restricting to "final" limit pseudocharacters); in the SL(2) case, this amounts to throwing out the nonspherical principal series with infinitesimal character zero since their constituents are already accounted for by the addition of the limits of discrete series as standard modules. We begin with the enlarged set of standard modules.

A *limit pseudocharacter* of H is a triple

$$\gamma = (\Psi, \Gamma, \bar{\gamma})$$

with the following properties:

(L1) Ψ is a positive system for the imaginary roots of $\mathfrak{h}$ in $\mathfrak{g}$, Γ is an irreducible representation of H; and $\bar{\gamma} \in \mathfrak{h}^*$;

(L2) If $\alpha \in \Psi$, then $\langle \bar{\gamma}, \alpha \rangle \geq 0$;

(L3) $d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi)$.

Let $\mathcal{P}_{\lim}(H)$ denote the limit pseudocharacters for H. Clearly $\mathcal{P}_{\mathrm{reg}}(H) \subset \mathcal{P}_{\lim}(H)$: the regular pseudocharacter $(\Gamma, \bar{\gamma})$ gets mapped to $(\Psi, \Gamma, \bar{\gamma})$ where Ψ is specified by (R2) above. But the notion of limit pseudocharacter allows for more: the infinitesimal character $\bar{\gamma}$ can now be singular according to (L2). In this case, Ψ is not uniquely specified of course; indeed (in the connected case) we should think of Ψ specifying a chamber of discrete series from which we translate to infinitesimal character $\bar{\gamma}$ of a limit of discrete series for L. To each $\gamma \in \mathcal{P}_{\lim}(H)$, we may define a standard module $X(\gamma)$ as above; this time X^L is a limit of discrete series and the choice of N (as in (4)) is a little messier. Two issues present themselves: the limit of discrete series X^L may be zero; and (as in the case of $\mathrm{SL}(2,\mathbb{R})$), we need to rule out certain reducibilities among the standard modules $X(\gamma)$. The conditions (F1) and (F2) below are designed with these respective issues in mind, and determine the standard modules we want to throw out.

A limit pseudocharacter is *final* if

(F1) If α is a simple root in Ψ such that $\langle \alpha, \bar{\gamma} \rangle = 0$, then α is noncompact;

(F2) If α is a real root of $\mathfrak{h}$ in $\mathfrak{g}$ and $\langle \alpha, \bar{\gamma} \rangle = 0$, then α does not give reducibility; i.e. α does not satisfy the Speh-Vogan parity condition.

We write $\mathcal{P}_{\mathrm{fin}}(H)$ for the set of final limit pseudocharacters for H.

As alluded to above, to each $\gamma \in \mathcal{P}_{\mathrm{fin}}(H)$, we attach a standard module $X(\gamma)$; it is induced from a cuspidal parabolic MAN and a limit of discrete series on M[1]. Condition (F1) guarantees that $X(\gamma)$ is not zero.

Definition 5 Define $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}(H)$ to be the subset of $\gamma \in \mathcal{P}_{\mathrm{fin}}(H)$ such that the restriction of $\bar{\gamma}$ to $\mathfrak{a}$ is identically zero. We say that $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}(H)$ is the set of *tempered final limit pseudocharacters with real infinitesimal character.* The terminology is explained by Proposition 6 below.

For $\gamma \in \mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}(H)$ we let $\overline{X}(\gamma)$ denote the unique LKT constituent of the standard module $X(\gamma)$. (The unicity of such a constituent is not obvious, but it is true using arguments that lead to Theorem 1(1) above. Also note that this definition of $\overline{X}(\gamma)$ finesses the choice of N remarked upon in the footnote.) Here is a special case of the Langlands-Knapp-Zuckermann classification (with some elaboration by Vogan to handle all groups in Harish-Chandra's class).

Proposition 6 *Fix $\gamma_i \in \mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}(H_i)$. Then $\overline{X}(\gamma_1) \simeq \overline{X}(\gamma_2)$ if and only if (γ_1, H_1) is conjugate to (γ_2, H_2) by an element of K. Moreover, an irreducible tempered representation with real infinitesimal character is of the form $\overline{X}(\gamma_i)$.*

Of course the identical statement holds for all admissible representations and all final limit characters. We have not stated that since we haven't defined $\overline{X}(\gamma)$ precisely in that context; see [Vunit, Section 2].

Corollary 7 *The set of equivalence classed of irreducible admissible representations of K is parametrized by K orbits on $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}(H)$ (Definition 5). The parametrization takes an orbit $K \cdot (\gamma, H)$ in $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}(H)$ to the lowest K-type of $\overline{X}(\gamma)$.*

The corollary follows immediately from Proposition 6 and Theorem 1.

As promised, we conclude by describing how to compute LKTs in terms of this parametrization. Suppose X is any irreducible admissible representation of G. Using the classification above, we may write $X = \overline{X}(\gamma)$ for some $\gamma \in \mathcal{P}_{\mathrm{fin}}$. Then X is the LKT constituent of $X(\gamma)$. Now modify γ by making the e^{ν} factor trivial: that is, change $\bar{\gamma}$ to $\bar{\gamma}' := \bar{\gamma} - \nu$, let $\Gamma' := \Gamma \otimes e^{-\nu}$, but leave Ψ unchanged. The resulting γ' is still a limit pseudocharacter which satisfies the first final condition (F1). Unfortunately since we have changed the infinitesimal

[1] The choice of N requires care for some applications – does one want $X(\gamma)$ to have a unique irreducible quotient or a unique irreducible submodule? Different N change the composition series — but not the composition factors — of the induced module.

character, (F2) can fail. If (F2) does hold, then $\gamma' \in \mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}$, and the LKT of X is simply the one parametrized (according to Corollary 7) by γ'; in particular the LKT is unique. In the case that (F2) fails, the roots for which it fails define a maximal sequence of (generally multivalued) inverse Cayley transforms which we can apply to γ'. The result is a list $\gamma'_1, \ldots, \gamma'_k \in \mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}$. Then the LKTs of X are precisely those parametrized by $\gamma'_1, \ldots, \gamma'_k$. This completes the calculation of the LKT of X in parametrization of Corollary 7.

References

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[Vunit] D. Vogan, *Unitarizability of certain series of representations*, Ann. Math. **120**(1984), 141–187.