Real reductive subgroups of equal rank

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1 Introduction

Our motivation is the following problem:

Problem 1.1. *Let $G_{\mathbb{R}}$ be a real reductive group. Describe all real reductive subgroups $H_{\mathbb{R}} \subset G_{\mathbb{R}}$ of the same rank.*

We approach this problem from a more general setup. Let G be a reductive algebraic group. Recall that we have an exact sequence

$$1 \longrightarrow Int(G) \longrightarrow Aut(G) \longrightarrow Out(G) \longrightarrow 1.$$

A real form of G (up to equivalence) corresponds to an involution $\theta \in Aut(G)$ (up to conjugation). The real form $G_{\mathbb{R}}$ corresponding to θ has a maximal compact subgroup $K_{\mathbb{R}}$ with complexification $K = G^{\theta}$. Two real forms are in the same inner class if the corresponding involutions have the same image in $Out(G)$. For more precise definitions see an expository paper "Strong real forms and the Kac classification" by Jeffrey Adams (http://www.liegroups.org/papers/realforms.pdf).

We fix an inner class, i.e. an involution $\sigma \in Out(G)$. Let $G^{\Gamma} = G \rtimes \Gamma$ where $\Gamma = \{1, \sigma\} \cong \mathbb{Z}_2$. Let $Z(G)$ be the center of G.

Definition 1.2. *A strong real form of G is an element $x \in G^{\Gamma}$, such that $x \notin G$ and $x^2 \in Z(G)$.*

The adjoint action of such element x is an involution on G and we will denote it by θ_x. It is a complexified Cartan involution for the corresponding real form.

We are looking now for connected, equal rank subgroups $H \subset G$ which are θ_x−stable. Every such subgroup H has to contain a

θ_x−stable Cartan subgroup T. In fact, we can choose T to be the fundamental (i.e. most compact) Cartan subgroup in H. Note that such T might not be fundamental in G.

Therefore we are looking for a θ_x stable Cartan subgroup $T \subset G$ and a collection $\{\alpha_1, \dots, \alpha_m\}$ of roots for T in G. We require that all the roots $\alpha_1, \dots, \alpha_m$ are simple for H and that θ_x permutes the α_i's .

2 Easy case

We consider the case when $rk(H \cap K) = rk(H)$, where K is the maximal compact subgroup of G. Then T is the most compact Cartan subgroup of G. (Note: in the Atlas output such Cartan subgroup is always listed first with a number "0").

We are looking for a set $\{\alpha_1, \dots, \alpha_m\}$ of roots of T in G that correspond to a simple root system of type H. We also need to take care that the labelling of the roots in G matches the labelling of the roots of the group of the real points $H_{\mathbb{R}}$.

Example: possible complication. Suppose we would like to find $SU(2,1)$ in the split G_2. The software `atlas` labels simple roots for $SU(2,1)$ as "$n-n$" (n stands for "non compact imaginary"). Therefore we seek two roots in G_2 that generate A_2 and that are labelled "$n-n$". The problem is that these roots are not necessarily simple in G_2.

Further restrictions. We will assume additionally that the Dynkin diagram of our maximal proper reductive subgroup H can be obtained from the extended Dynkin diagram for G by removing one vertex. Now, if $\Delta^+(G,T)$ is a set of positive roots for G and $\{\alpha_1, \dots, \alpha_m\}$ are simple for our group $H \subset G$ then there exists an element w of the Weyl group $W(G,T)$ such that

$$\{w\alpha_1, \dots, w\alpha_m\} \subset \{ \text{ simple roots of } G\} \cup \{\text{lowest root}\}.$$

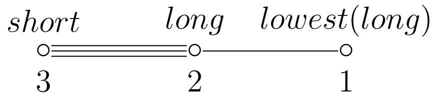
Warning: It is not true that every maximal equal rank proper reductive subgroup of G can be obtained from the extended Dynkin diagram for G by removing a single vertex. As an example consider $GL(2) \subset SL(3)$.

Hypothesis 1: If we delete one vertex in the extended Dynkin diagram of a group G that has a prime label p, then what's left is the diagram of a maximal proper equal rank reductive subgroup.

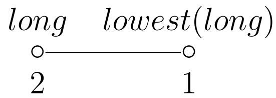
Hypothesis 2: If we delete two vertices with labels 1 in the extended diagram we obtain a diagram of a maximal Levi subgroup that is also a maximal proper equal rank reductive subgroup.

Hypothesis 3: These two methods give all maximal equal rank proper reductive subgroups of the group G.

Back to our example. Let's look at the extended Dynkin diagram for G_2 :



We obtain a Dynkin diagram of type A_2 after removing the vertex that has label 3:



Therefore, if we consider an arbitrary positive root system $\Delta^+(G_2, T)$, then the subgroup that corresponds to the set of roots { "long simple", "lowest long" } is an equal rank form of $SU(3)$.

The next question is, which form of $SU(3)$ did we obtain? To answer this, we need to compute the label for the lowest root. First we assign a value in $\mathbb{Z}/2\mathbb{Z}$ for each label: 1 for n ("non compact imaginary") and 0 for c ("compact"). Then we calculate the label for the lowest root according to the formula

$$\text{label of lowest root} \;=\; \sum_{\alpha \text{ simple}} m_\alpha \cdot label(\alpha) \mod 2,$$

where m_α is the multiplicity of a root α. In our case we get that

$$\text{label of lowest root} \;=\; \text{label of the short simple root}\ .$$

Therefore a Borel subgroup in G_2 labelled $x \equiv y$ gives an A_2 subsystem labelled $y - x$.

We look now at the `kgb` output for $SU(2,1)$:

```
0: 0 0 [n,n]
```

```
1: 0 0 [n,c]
2: 0 0 [c,n],
```

and for $SU(3)$:

```
0: 0 0 [c,c].
```

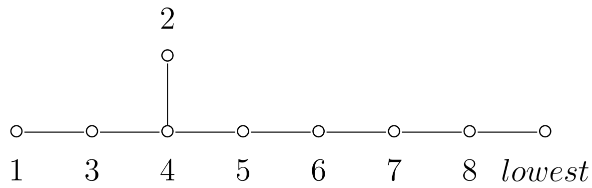
We notice that only the labels of the former appear in the `kgb` output for G_2 :

```
0: 0 0 [n,n]
1: 0 0 [n,c]
2: 0 0 [c,n]
3: 0 0 [r,C]
4: 0 0 [C,r]
5: 0 0 [C,C]
6: 0 0 [C,C]
7: 0 0 [C,n]
8: 0 0 [n,C]
9: 0 0 [r,r]
```

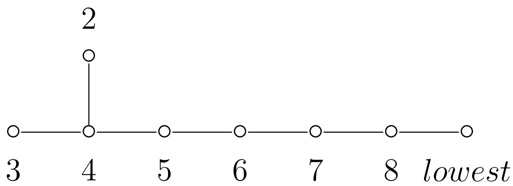
Moreover, they appear exactly once. The conclusion is that we have exactly one conjugacy class of $SU(2,1)$ in the split G_2.

3 Another example: SO(12,4) and split E_8

This time we search for $SO(12,4)$ inside the split E_8. We start with the extended Dynkin diagram for E_8(the numbering of the roots corresponds to the `atlas` numbering):



We find D_8 by deleting the vertex numbered "1":

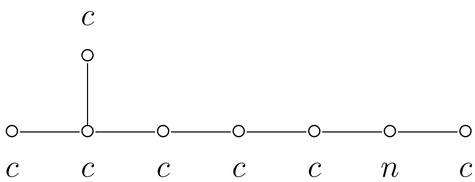


Now we would like to know if indeed we obtained $SO(12,4)$. Again, we will need a formula for the label of the lowest root. We get that

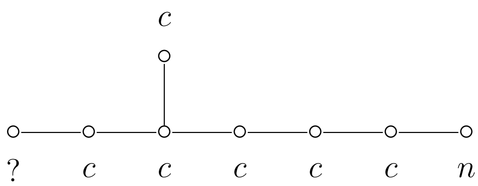
$$\text{label of lowest root} \;=\; \sum_{\alpha \text{ simple}} m_\alpha \cdot \text{label}(\alpha) \mod 2$$

$$= \text{label}(2) + \text{label}(5) + \text{label}(7) \mod 2.$$

As before, 1 corresponds to label n and 0 corresponds to label c. Next we choose a Borel subgroup in $SO(12,4)$ that has the following labels



If the form we obtained was $SO(12,4)$ then there would exits a Borel subgroup in E_8 with the labels



On the other hand, such a labelling would give us an $SO(12,4)$, since the label for the lowest root (according to our formula) equals to

$$\text{label}(2) + \text{label}(5) + \text{label}(7) = c + c + c = c \mod 2,$$

and that is compatible with our choice.

Therefore looking for $SO(12,4)$ is equivalent to looking for a string [?,c,c,c,c,c,c,n] in the `kgb` output for (split) E_8. Since we did not find such a string, we conclude that there are no copies of $SO(12,4)$.