Multiplicities of $K$-types in principal series

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joint work with Dan Barbasch

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INTRODUCTION/MOTIVATION
find the unitary dual of split $G_R$ → discuss unitarity of Langlands quotients of principal series

$J_P(\delta, \nu) \sim P = MAN$ → signature of some Hermitian operators $A_\mu(\delta, \nu)$

$\mu \in \widehat{K}, \delta \in \widehat{M}, \nu \in a^*_C$

The intertwining operator $A_\mu(\delta, \nu)$ acts on $\text{Hom}_M(\delta, \mu)$.

**PROBLEM** Understand the representation of $W(\delta)$ (the stabilizer of $\delta$ in $W$) on the space $\text{Hom}_M(\delta, \mu), \forall \delta \in \widehat{M}, \mu \in \widehat{K}$. 
Spherical unitary dual

Use spherical petite K-types to prove that $J(\nu)_\mathbb{R}$ unit. $\Rightarrow$ $J(\nu)_{\mathbb{Q}_p}$ unit.

$\text{Barbasch-Vogan}$

Spherical unitary dual of split $G(\mathbb{R})$

$?$

$\uparrow$

candidates: $J(\nu)_{\mathbb{R}}$

$J(\nu)_{\mathbb{R}}$ unitary $\iff$

$A_\mu(\nu) \geq 0$, $\forall \mu \in \hat{K}$

Spherical unitary dual of split $G(\mathbb{Q}_p)$

$\checkmark$

$\uparrow$

candidates: $J(\nu)_{\mathbb{Q}_p}$

$J(\nu)_{\mathbb{Q}_p}$ unitary $\iff$

$A_\psi(\nu) \geq 0$, $\forall \psi \in \hat{W}_{relev}$
Non-spherical unitary dual

non-spher. unitary dual of split $G_\mathbb{R}$

use non-spherical petite K-types to investigate whether $J^G(\delta, \nu)$ unit $\Rightarrow J^{G_0(\delta)}(\nu_0)$ unit

Barbasch–Pantano

candidates: $J^G(\delta, \nu)$ $\leadsto$ define $G^{\delta}$

spherical unitary dual of split $G_0(\delta)$

candidates: $J^{G_0^\delta}(\nu_0)$

$J^G(\delta, \nu)$ unitary

$\Leftrightarrow A_\mu(\delta, \nu) \geq 0$ $\forall \mu \in \hat{K}$

$\Rightarrow$ $\text{Hom}_M(\delta, \nu)$ $\Leftarrow$

$J^{G_0^\delta}(\nu_0)$ unitary

$\Leftrightarrow A_\psi(\nu) \geq 0$ $\forall \psi \in \hat{W}_0$ relevant
Two projects

**BIG PROJECT**

Find an inductive algorithm to compute the $W(\delta)$-representation

$\text{Hom}_M(\delta, \mu)$

→ July

**SMALL PROJECT**

Find an inductive algorithm to compute

$\text{dim}[\text{Hom}_M(\delta, \mu)]$

→ today
Plan of the talk

- Standard Notation
- Multiplicities of $K$-types in principal series
- Some easy examples (linear case)
- Non-linear case (what we know...)
- An inductive algorithm to compute multiplicities
- Generalization
PART 1

- **Standard Notation**

- Multiplicities of $K$-types in principal series

- Some easy examples (*linear case*)

- Non-linear case (*what we know...*)

- An inductive algorithm to compute multiplicities

- Generalization
Notation

- **G** a real reductive Lie group ← *split group*
- **K** the maximal *compact* subgroup of **G**
- **K-types** the irreducible representations of **K**
  \[ \mu = \sum a_j \omega_j \text{, with } a_j \geq 0 \text{ and } \omega \text{ fundamental} \]
- **θ** a Cartan involution on **g**
- **g = ℱ ⊕ ℓ** the Cartan decomposition of **g**
- **a** a maximal abelian subspace of **p**, \( A = \exp(\mathfrak{a}) \)
- **M = Z_K(\mathfrak{a})** ← *finite subgroup of K*
- **P = MAN** a minimal parabolic subgroup of **G**
Minimal Principal Series

\[ \begin{align*}
P = MAN & \quad \text{minimal parabolic subgroup of } G \\
(\delta, V^{\delta}) & \quad \text{irreducible representation of } M \\
\nu : \mathfrak{a} \to \mathbb{C} & \quad \text{dominant character of } A
\end{align*} \]

principal series \[ I_P(\delta, \nu) = \text{Ind}^G_{MAN}(\delta \otimes \nu \otimes \text{triv}) \]

\(G\) acts by left translation on the space of functions

\[ \{ F : G \to V^{\delta} : F |_K \in L^2, F(x\text{man}) = e^{-(\nu+\rho)\log(a)}\delta(m)^{-1}F(x), \ \forall \text{man} \in P \} \]
PART 2

- Standard Notation

- Multiplicities of $K$-types in principal series

- Some easy examples (linear case)

- Non-linear case (what we know...)

- An inductive algorithm to compute multiplicities

- Generalization
Multiplicities of $K$-types in Principal Series

Which irreducible representations $\mu$ of $K$ appear in the principal series $I_P(\delta, \nu)$, and with what multiplicities?
The multiplicity of a $K$-type $\mu$ in $I_P(\delta, \nu)$ is defined by

$$m(\mu, I_P(\delta, \nu)) = \dim [\text{Hom}_K(\mu, \text{Res}_K I_P(\delta, \nu))]$$

By Frobenius reciprocity, it is independent of the parameter $\nu$:

$$m(\mu, I_P(\delta, \nu)) = m(\delta, \mu) = \dim [\text{Hom}_M(\delta, \text{Res}_M \mu)]$$

$\Rightarrow$ We need to study the restriction of $K$-types to $M$.
PART 3

- Standard Notation
- Multiplicities of $K$-types in principal series
  
  - Some easy examples (linear case)
  
- Non-linear case (what we know...)
- An inductive algorithm to compute multiplicities
- Generalization
The example of $SL(2, \mathbb{R})$

- $G = SL(2, \mathbb{R})$, $K = SO(2, \mathbb{R})$, $M = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \simeq \mathbb{Z}_2$

- $\hat{K} = \mathbb{Z}$, $\hat{M} = \{\text{trivial, sign}\}$

- $\text{Res}_M(\mu_n) = \begin{cases} \text{trivial} & \text{if } n \text{ is even} \\ \text{sign} & \text{if } n \text{ is odd} \end{cases}$

$$\Rightarrow m(\mu_{2l}, I_P(\delta, \nu)) = \begin{cases} 1 & \text{if } \delta \text{ is trivial} \\ 0 & \text{if } \delta \text{ is sign} \end{cases}$$

and $$m(\mu_{2l+1}, I_P(\delta, \nu)) = \begin{cases} 0 & \text{if } \delta \text{ is trivial} \\ 1 & \text{if } \delta \text{ is sign} \end{cases}$$
The example of $SL(3, \mathbb{R})$

- $G = SL(3, \mathbb{R}), K = SO(3, \mathbb{R})$
- $M = \{\text{diag}(\epsilon_1, \epsilon_2, \epsilon_3): \epsilon_i = \pm 1, \Pi \epsilon_i = 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\widehat{K} = \{\mathcal{H}_n\}_{n \in \mathbb{N}} = \{p(x, y, z): \text{harmonic, homog. of degree } n\}$
- $\widehat{M} = \{\text{triv} \otimes \text{triv}, \text{triv} \otimes \text{sign}, \text{sign} \otimes \text{triv}, \text{sign} \otimes \text{sign}\}$
- $\mathcal{H}_{2l} \mid_M = [tr \otimes tr]^{l+1} \oplus [tr \otimes \text{sign}]^l \oplus [\text{sign} \otimes tr]^l \oplus [\text{sign} \otimes \text{sign}]^l$

\[ m(\mathcal{H}_{2l}, I_P(\delta, \nu)) = \begin{cases} 
  l + 1 & \text{if } \delta = tr \otimes tr \\
  l & \text{otherwise}
\end{cases} \]

There are similar formulas for $\mathcal{H}_{2l+1}$
Suppose that

- $\mathbb{G}$: a simple, connected and simply connected real reductive algebraic group
- $G$: the split real form of $\mathbb{G}$
- $\mathring{G}$: the (unique) two-fold cover of $G$

then

\[
\mathring{G} \text{ is non-linear and } \mathring{M} \text{ is non-abelian}
\]
• Standard Notation

• Multiplicities of $K$-types in principal series

• Some easy examples (*linear case*)

  • **Non-linear case** (*what we know about $\widetilde{M}$...*)

• An inductive algorithm to compute multiplicities

• Generalization
For each root $\alpha$, we can choose a Lie algebra homomorphism

$$\phi_\alpha : sl(2, \mathbb{R}) \to g$$

such that

$$Z_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in t = \text{Lie}(K).$$

Exponentiating $\phi_\alpha$, we obtain

$$\Phi_\alpha : SL(2, \mathbb{R}) \to G \quad \tilde{\Phi}_\alpha : \tilde{SL}(2, \mathbb{R}) \to \tilde{G}.$$  

**Definition:** $\alpha$ is **metaplectic** if $\tilde{\Phi}_\alpha$ does not factor to $SL(2, \mathbb{R})$.

If $G$ is not of type $G_2$, then **metaplectic $\Leftrightarrow$ long**, if $G$ is of type $G_2$, then all roots are metaplectic.
More notation: \( \tilde{m}_\alpha = \exp_{\tilde{G}}(\pi Z_\alpha) \) and \( m_\alpha = \exp_G(\pi Z_\alpha) \)
**Structure of \(\tilde{M}\)**

- **GENERATORS:** \(\{\tilde{m}_\alpha\}_{\alpha \text{ simple}}\)

- **RELATIONS:** \(\tilde{m}_\alpha^2 = \begin{cases} -I & \text{if } \alpha \text{ is metaplectic} \\ +I & \text{otherwise} \end{cases} \)

  and \(\{\tilde{m}_\alpha, \tilde{m}_\beta\} = \begin{cases} (-I)^{\langle\alpha, \beta\rangle} & \text{if } \alpha \text{ and } \beta \text{ are both metaplectic} \\ +I & \text{otherwise.} \end{cases} \)

- **ELEMENTS:** Choose an ordering of the simple roots. Every element of \(\tilde{M}\) can be written uniquely in the form

  \[ \varepsilon \tilde{m}_{\alpha_1}^{n_1} \tilde{m}_{\alpha_2}^{n_2} \cdots \tilde{m}_{\alpha_r}^{n_r} \]

  with \(\varepsilon = \pm 1\), and \(n_j = 0 \text{ or } 1\).
Example: $\tilde{M} \subset \tilde{E}_6$

**GENERATORS:** $\{\tilde{m}_{\alpha_i}\}_{i=1...6}$

**RELATIONS:** $\tilde{m}_{\alpha_i}^2 = -I$ for all $i = 1 \ldots 6$, and

$$\{\tilde{m}_{\alpha_i}, \tilde{m}_{\alpha_j}\} = (-I)^{\langle \alpha_i, \alpha_j \rangle} = \begin{cases} (-I) & \text{if } \alpha_i \text{ and } \alpha_j \text{ are adjacent} \\ (+I) & \text{otherwise.} \end{cases}$$

**CENTER:** $Z(\tilde{M}) = \{\pm I\} \cong \mathbb{Z}_2$
Example: $\tilde{M} \subset \tilde{F}_4$

**GENERATORS:** $\{\tilde{m}_{\alpha_i}\}_{i=1}^{4}$

**RELATIONS:** $\tilde{m}_{\alpha}^2 = \begin{cases} -I & \text{if } \alpha \text{ is long} \\ +I & \text{if } \alpha \text{ is short} \end{cases}$

and $\{\tilde{m}_{\alpha}, \tilde{m}_{\beta}\} = \begin{cases} (-I) & \text{if } \alpha \text{ and } \beta \text{ are both long} \\ (+I) & \text{otherwise.} \end{cases}$

**CENTER:** $Z(\tilde{M}) = \langle -I, \tilde{m}_{\alpha_3}, \tilde{m}_{\alpha_4} \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Representations of $\widetilde{M}$

$\widetilde{M}$ is a cover of the abelian group $M$. There is an exact sequence

$$1 \to \{\pm I\} \to \widetilde{M} \to M \to 1.$$ 

A repr. of $\widetilde{M}$ is called genuine if $(-I)$ does not act trivially.

- The non-genuine representations of $\widetilde{M}$ have dim. 1. They are determined by the value of $\delta(\tilde{m}_{\alpha_i}) = \pm 1$.
- The genuine repr.s of $\widetilde{M}$ have dim. $n = |\widetilde{M}/Z(\widetilde{M})|^{\frac{1}{2}}$. They are determined by the restriction to $Z(M)$.

$$\{\text{genuine repr.s of } \widetilde{M}\} \leftrightarrow \{\text{genuine characters of } Z(\widetilde{M})\}$$

$$\delta \rightarrow \lambda \text{ s.t. } \text{Res } \delta = \lambda^{\oplus n}$$

$$\delta \text{ s.t. } \text{Ind } \lambda = \pi^{\oplus n} \leftrightarrow \lambda$$
Every non-genuine representation is one-dimensional, and is determined by the 6-upla \([\delta(\tilde{m}_{\alpha_1}), \ldots, \delta(\tilde{m}_{\alpha_6})]\). For \(\delta(\tilde{m}_{\alpha_i}) = \pm 1\), there are \(2^6\) distinct non-genuine representations.

The group \(Z(\tilde{M})\) has one genuine repr. \(\chi_g\), given by \(\chi_g(-I) = -1\). Hence \(\tilde{M}\) has only one genuine repr. \(\delta_g\). The dimension of \(\delta_g\) is

\[
|\tilde{M}/Z(\tilde{M})|^{\frac{1}{2}} = \sqrt{2 \cdot 2^6 / 2} = 8.
\]

To compute the character of \(\delta_g\), we use the fact \(8\delta_g = \text{Ind}_{Z(\tilde{M})}^{\tilde{M}} \chi_g\).
PART 5

- Standard Notation
- Multiplicities of $K$-types in principal series
- Some easy examples (linear case)
- Non-linear case (what we know...)

- An inductive algorithm to compute multiplicities
- Generalization
An inductive algorithm to compute multiplicities

**INPUT**

- tensor product of $W$-orbits of $\tilde{M}$-types
- restriction to $\tilde{M}$ of fundamental $\tilde{K}$-types

**OUTPUT**

- restriction to $\tilde{M}$ of every other $\tilde{K}$-type
“essentially” recovered from $\bigotimes$ of fine $\tilde{K}$-types

tensor product of $W$-orbits of $\tilde{M}$-types

restriction to $\tilde{M}$ of fundamental $\tilde{K}$-types

computed by hand

multiplicities of $\tilde{K}$-types in principal series

restriction to $\tilde{M}$ of every other $\tilde{K}$-type

A VERY COOL FACT: in order to restrict $\tilde{K}$-types to $\tilde{M}$, we need very little information about the actual repr.s of $\tilde{M}$
Computing the restriction of a $\tilde{K}$-type $\mu$ to $\tilde{M}$

(by induction on level and lexicographical order)

- $\mu$ embeds in a tensor product of fundamental representations
- we can write $\mu = \mu' + \omega$, with $\omega$ fundamental and $\mu'$ lower in the induction

\[ \mu' \otimes \omega = \mu + (\text{lower terms}) \]  

\[ (\star) \]

- The restriction of $\mu'$ and $\omega$ to $\tilde{M}$ are known (by induction)
- The restriction of $\mu' \otimes \omega$ to $\tilde{M}$ is computed using the table of tensor product of $W$-orbits of $\tilde{M}$-types (base of induction)
- Equation $(\star)$ gives $\text{Res}_{\tilde{M}} \mu$ (by comparison)
An example

Let $\tilde{G} = \tilde{F}_4$, $\tilde{K} = SP(1) \times SP(3)$ and $\mu = (0|200)$.

$$(0|200) = (0|100) + (0|100) \quad \Rightarrow \quad \mu \mapsto \mu' \otimes \omega$$

$\mu$ lower in induction
$\mu'$ fundamental

Restriction to $\tilde{M}$ gives:

$$(0|100) \otimes (0|100) = (0|200) \oplus (0|110) \oplus (0|00)$$

We know that $\bar{d}_6 \otimes \bar{d}_6 = 3\delta_0 \oplus 3\bar{d}_3 \oplus 2\bar{d}_{12}$. Hence

$$\text{Res}(0|200) = 3\bar{d}_3 \oplus \bar{d}_{12}$$

by comparison.
BASE OF INDUCTION
for double covers of exceptional groups
The two-fold cover of $E_6$

- $\tilde{G} = \tilde{E}_6$
- $\tilde{K} = Sp(4)$

<table>
<thead>
<tr>
<th>$W$-orbit of $\tilde{M}$-types</th>
<th>dim.</th>
<th>fine $\tilde{K}$-type</th>
<th>$W_\delta^0$</th>
<th>$W(\delta)$</th>
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<td>$\delta_1$</td>
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The two-fold cover of $E_8$

- $\tilde{G} = \tilde{E}_8$
- $\tilde{K} = Spin(16)$

<table>
<thead>
<tr>
<th>$W$-orbit of $\tilde{M}$-types</th>
<th>dim.</th>
<th>fine $\tilde{K}$-type</th>
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<td>$135\delta_0 + 72\bar{\delta}_{120}$</td>
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The two-fold cover of $F_4$

- $\tilde{G} = \tilde{F}_4$
- $\tilde{K} = Sp(1) \times Sp(3)$

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<th>$W$-orbit of $\tilde{M}$-types</th>
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<th>\tilde{\delta}_3</th>
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<td>\tilde{\delta}_{12}</td>
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<td>\tilde{\delta}_{12}</td>
<td>3\tilde{\delta}_6</td>
<td>3\delta_0 + 3\tilde{\delta}<em>3 + 2\tilde{\delta}</em>{12}</td>
<td>12\delta_2 + 8\tilde{\delta}_6</td>
</tr>
<tr>
<td>\tilde{\delta}_{12}</td>
<td>4 \tilde{\delta}_6</td>
<td>3\tilde{\delta}_{12}</td>
<td>12\delta_2 + 8\tilde{\delta}_6</td>
<td>12\delta_0 + 12\tilde{\delta}<em>3 + 8\tilde{\delta}</em>{12}</td>
</tr>
</tbody>
</table>
The two-fold cover of $E_7$

- $\tilde{G} = \tilde{E}_7$
- $\tilde{K} = SU(8)$

<table>
<thead>
<tr>
<th>$W$-orbit of $\tilde{M}$-types</th>
<th>dim.</th>
<th>fine $\tilde{K}$-type</th>
<th>$W^0_\delta$</th>
<th>$W(\delta)$</th>
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<tr>
<td>$\delta_1$</td>
<td>1</td>
<td>$(0)$</td>
<td>$W(E_7)$</td>
<td>$W(E_7)$</td>
</tr>
<tr>
<td>$\delta_8$</td>
<td>8</td>
<td>$w_1$</td>
<td>$W(E_7)$</td>
<td>$W(E_7)$</td>
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<tr>
<td>$\delta_8^*$</td>
<td>8</td>
<td>$w_7$</td>
<td>$W(E_7)$</td>
<td>$W(E_7)$</td>
</tr>
<tr>
<td>$\tilde{\delta}_{28}$</td>
<td>$28 \cdot 1$</td>
<td>$w_2, w_6$</td>
<td>$W(E_6)$</td>
<td>$W(E_6) \rtimes \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\tilde{\delta}_{36}$</td>
<td>$36 \cdot 1$</td>
<td>$2w_1, 2w_7$</td>
<td>$W(A_7)$</td>
<td>$W(A_7) \rtimes \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\tilde{\delta}_{63}$</td>
<td>$63 \cdot 1$</td>
<td>$w_1 + w_7$</td>
<td>$W(D_6A_1)$</td>
<td>$W(D_6A_1)$</td>
</tr>
<tr>
<td>fundamental $\tilde{K}$-types</td>
<td>$#\delta_1$</td>
<td>$#\tilde{\delta}_{28}$</td>
<td>$#\tilde{\delta}_{36}$</td>
<td>$#\tilde{\delta}_{63}$</td>
</tr>
<tr>
<td>-----------------------------</td>
<td>-------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
</tr>
<tr>
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<td>0</td>
</tr>
<tr>
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<td>0</td>
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<tr>
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<td>0</td>
</tr>
<tr>
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<td>0</td>
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<tr>
<td></td>
<td>$\delta_8$</td>
<td>$\delta^*_8$</td>
<td>$\bar{\delta}_{28}$</td>
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<tr>
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<td>$\bar{\delta}<em>{28} + \bar{\delta}</em>{36}$</td>
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<td>$28\delta^*_8$</td>
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<td>$\bar{\delta}<em>{28} + \bar{\delta}</em>{36}$</td>
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<tr>
<td>$\bar{\delta}_{28}$</td>
<td>$28\delta^*_8$</td>
<td>$28\delta_8$</td>
<td>$28\delta_1 + 12\bar{\delta}_{63}$</td>
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<tr>
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<td>$36\delta^*_8$</td>
<td>$36\delta_8$</td>
<td>$16\bar{\delta}_{63}$</td>
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<td>$63\delta_8$</td>
<td>$63\delta^*_8$</td>
<td>$27\bar{\delta}<em>{28} + 28\bar{\delta}</em>{36}$</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$\bar{\delta}_{36}$</th>
<th>$\bar{\delta}_{63}$</th>
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</thead>
<tbody>
<tr>
<td>$\delta_8$</td>
<td>$36\delta^*_8$</td>
<td>$63\delta_8$</td>
</tr>
<tr>
<td>$\delta^*_8$</td>
<td>$36\delta_8$</td>
<td>$63\delta^*_8$</td>
</tr>
<tr>
<td>$\bar{\delta}_{28}$</td>
<td>$16\bar{\delta}_{63}$</td>
<td>$27\bar{\delta}<em>{28} + 28\bar{\delta}</em>{36}$</td>
</tr>
<tr>
<td>$\bar{\delta}_{36}$</td>
<td>$36\delta_1 + 20\bar{\delta}_{63}$</td>
<td>$36\bar{\delta}<em>{28} + 35\bar{\delta}</em>{36}$</td>
</tr>
<tr>
<td>$\bar{\delta}_{63}$</td>
<td>$36\bar{\delta}<em>{28} + 35\bar{\delta}</em>{36}$</td>
<td>$63\delta_1 + 62\bar{\delta}_{63}$</td>
</tr>
</tbody>
</table>
Restriction to $\tilde{M}$ of the fundamental $\tilde{K}$-types

the example of $\tilde{E}_6$

$\tilde{G} = \tilde{E}_6$
$\tilde{K} = Sp(4)$

Fundamental $\tilde{K}$-types: $w_1, w_2, w_3, w_4$

$W$-orbits of $\tilde{M}$-types: $\delta_1, \delta_8, \delta_{27}, \text{and } \delta_{36}$

- $\text{Res}_{\tilde{M}} w_1 = \delta_8$, and $\text{Res}_{\tilde{M}} w_2 = \delta_{27}$ (fine $\tilde{K}$-types)
- $w_3$ is genuine, and has dimension 48, hence $\text{Res}(w_3) = 6\delta_8$
- $(w_4)^{\tilde{M}}$ is the reflection repr. 6$^p$, because $w_4$ is the repr. of $\tilde{K}$ on $\mathfrak{p}$. For dimensional reasons, $\text{Res}(w_4) = 6\delta_1 \oplus \delta_{36}$. 
some examples for $\tilde{E}_6$

- $\delta_8 \otimes \delta_8 = \text{Res}_M[w_1 \otimes w_1] = \text{Res}_M[(0) \oplus w_2 \oplus 2w_1] = \delta_1 \oplus \delta_{27} \oplus \delta_{36}$
- $\tilde{\delta}_{36} \otimes \tilde{\delta}_{36} = \text{Res}_M[(2w_1) \otimes (2w_1)] = \text{Res}_M[(0) \oplus w_2 \oplus (2w_1)] \oplus \text{Res}_M[(2w_2) \oplus (2w_1 + w_2) \oplus (4w_1)]$

First, we compute $(2w_2)^\tilde{M}$. Because $(2w_2) \hookrightarrow (w_2 \otimes w_2)$ and

$$(w_2 \otimes w_2)^\tilde{M} = \text{Ind}_{W(\delta_{27})}^{W(E_6)} \text{Hom}_\tilde{M}(\delta_{27}, w_2) = \text{Ind}_{W(D_5)}^{W(E_6)}(5|0)$$

we can write:

$$(2w_2)^\tilde{M} = (w_2 \otimes w_2)^\tilde{M} - (w_1 + w_3)^\tilde{M} - w_4^\tilde{M} - 0^\tilde{M} = 20_p.$$
Similarly, we find \((4w_1)\widetilde{M} = 15_q\). Then

\[
\text{Res}_{\widetilde{M}}(4w_1) = 15\delta_1 \oplus b\delta_{27} \oplus c\delta_{36}.
\]

Comparing dimensions, we find that \(35 = 3b + 4c\) hence \(c = 2, 5\) or \(8\). We also notice that \(c = \dim[\text{Hom}_{\widetilde{M}}(\delta_{36}, 4w_1)]\). Because

\[
\text{Ind}_{W(A_5 A_1)}^{W(E_6)} \text{Hom}_{\widetilde{M}}(\delta_{36}, 4w_1) = (2w_1 \otimes 4w_1)\widetilde{M} \subseteq (4w_1)\widetilde{M} = 15_q
\]

the \(W(A_5 A_1)\)-representation \(\text{Hom}_{\widetilde{M}}(\delta_{36}, 4w_1)\) is a submodule of

\[
\text{Res}_{W(A_5 A_1)}^{W(E_6)}[15_q] = [(33) \otimes (11)] \oplus [(42) \otimes (2)] \oplus [(6) \otimes (2)].
\]

Hence \(c = 5\), and \(\text{Res}_{\widetilde{M}}(4w_1) = 15\delta_1 \oplus 5\delta_{27} \oplus 5\delta_{36}\).

The restrictions of \((2w_1 + w_2)\) and \((2w_2)\) are computed similarly. Then

\[
\delta_{36} \otimes \delta_{36} = 36\delta_1 \oplus 20\delta_{27} \oplus 20\delta_{36}.
\]
PART 6

- Standard Notation

- Multiplicities of $K$-types in principal series

- Some easy examples (linear case)

- Non-linear case (what we know...)

- An inductive algorithm to compute multiplicities

- Generalization
An inductive algorithm to compute multiplicities (revisited)

**INPUT**

- Tensor product of orbits of $\tilde{M}$-types
- Restriction to $\tilde{M}$ of fundamental $\tilde{K}$-types

**OUTPUT**

- Dimension of $\text{Hom}_{\tilde{M}}(\delta, \mu)$

\[ \forall \tilde{M}\text{-type } \delta \text{ and } \forall \tilde{K}\text{-type } \mu \]
Generalization

**INPUT**

- Tensor product of orbits of $\tilde{M}$-types
- Restriction to $\tilde{M}$ of fundamental $\tilde{K}$-types
- $\text{Hom}_{\tilde{M}}(\delta, w \otimes \mu_\tau)$ for all fundamental $\tilde{K}$-type $w$ and all $\tilde{M}$-type $\delta, \tau$

**OUTPUT**

- $\text{Hom}_{\tilde{M}}(\delta, \mu)$ for all $\tilde{M}$-type $\delta$ and all $\tilde{K}$-type $\mu$
- As a $W(\delta)$-representation

**Double Stabilizer** $W(\delta, \tau)$
DETAILS

... coming soon...