SOFTWARE FOR COMPUTING STANDARD REPRESENTATIONS

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ABSTRACT. Let $G_{\mathbb{R}}$ be the real points of a complex connected reductive algebraic group $G$. Let $K_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathbb{R}}$. We describe an algorithm for computing restrictions of standard representations of $G_{\mathbb{R}}$ to $K_{\mathbb{R}}$. We are currently implementing the algorithm as a package of the Atlas of Lie Groups and Representations software developed by Fokko du Cloux.

1. Introduction

Let $\theta$ be a Cartan involution for $g_{\mathbb{R}} = \text{Lie}(G_{\mathbb{R}})$ then $\theta$ extends to an involution of $g = \text{Lie}(G)$ giving $g = \mathfrak{k} + \mathfrak{p}$, the usual Cartan decomposition with $+1$-eigenspace $\mathfrak{k}$ and $-1$-eigenspace $\mathfrak{p}$. Furthermore $\text{Lie}(K) = \mathfrak{k}$. Let $B$ be a Borel subgroup of $G$ and $H$ a Cartan subgroup of $G$ such that $H \subseteq B$. Then $K$ acts on the flag variety $G/B$. Let $H_1$ be a $\theta$-stable Cartan subgroup of $G$ defined by the positive root system $\Delta_{+1}^+$ with $B_1$, the Borel subgroup defined by $\Delta_{+1}^+$. The orbit $K.B_1 = \mathcal{O}_1$ in $G/B$ determines $H_1$ up to $K$-conjugacy. The underlying philosophy is that irreducible $G_{\mathbb{R}}$-representations are to be understood via the algebraic structure of their $(g, K_{\mathbb{R}})$ modules.

In Atlas $K$-orbits on $G/B$ correspond to

$$\coprod_{\text{disjoint union over } K\text{-classes of } \theta\text{-stable } H_i} W(G, H_i)/W(K, H_i).$$

A continued standard representation $(\Delta_{+1, \text{im}}^+, \lambda_1)$ is given by an orbit $K.B_1$ and a Harish-Chandra module $(\text{Lie}(H_1), H_1 \cap K)$.

Theorem 1.1. (Hecht, Milicic, Schmid, Wolf)

(i) Continued standard representations depend only on the imaginary positive roots $\Delta_{1, \text{im}}^+$.

(ii) If $\lambda_1$ is positive on $\Delta_{1, \text{im}}^+$ then $(\Delta_{1, \text{im}}^+, \lambda_1)$ is an actual standard representation.

(iii) Restriction to $K$ depends only on $\Delta_{1, \text{im}}^+$ and $\lambda_1|_{H_1 \cap K}$.

1.1. Example. Let $G_{\mathbb{R}} = U(2, 1)$ and $K_{\mathbb{R}} = U(2) \times U(1)$. Then $G = GL(3, \mathbb{C})$ and $K = GL(2, \mathbb{C}) \times GL(1, \mathbb{C})$.

There are two $K$-conjugacy classes of $\theta$-stable Cartan subgroups:

$H_1$ = diagonal matrices that is $(\mathbb{C}^\ast)$.³
and

\[
H_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cosh t & \sinh t \\
0 & \sinh t & \cosh t
\end{pmatrix}, \quad \begin{pmatrix}
e^\phi & 0 & 0 \\
0 & e^\gamma & 0 \\
0 & 0 & e^\gamma
\end{pmatrix}
\]

In the $H_1$ case $\Delta_1^+ = \Delta_1^+ \text{im}$ and $W(G, H_1) = S_3$. Let $\rho = (1, 0, -1)$. If we want to compute standard representations with infinitesimal character $\rho$ then each $\lambda$ will correspond to a permutation of $\rho$ with the first two coordinates in decreasing order.

So $\lambda_1$ takes the following values: $(1, 0, -1)$, $(1, -1, 0)$, and $(0, -1, 1)$ accounting for 3 discrete series of $U(2, 1)$.

In the case of $H_2$ there are no imaginary roots and $[W(K, H_2, R) = S_1 \times S_2]$. Here $\lambda_2$ takes the following values; $(1, 0, -1)$, $(0, 1, -1)$, and $(-1, 1, 0)$. When restricted to $K$ we obtain $\phi - \gamma, 0$ and $-\phi + \gamma$, which account for 3 principal series of $U(2, 1)$.

1.2. How to describe restriction to $K$. First we will work on the fundamental Cartan in order to obtain a simpler formula. We note that $K.B_1 \cong K/(B_1 \cap K)$ which is a complete flag variety for $K$ and up to a $\rho$-shift $\lambda_1$ corresponds to a line bundle $\mathcal{L}_{\lambda_1}$ on $K/(B_1 \cap K)$. This is data for discrete series of $G_\mathbb{R}$. The lowest $K$-type is $H^{top}(K/(B_1 \cap K), \mathcal{L}_{\lambda_1})$ which corresponds to an irreducible representation of $K$ with highest weight $\lambda_1$ (up to a $\rho_G - 2\rho_K$)-shift. Here $top = \dim_K K/(B_1 \cap K)$.

Ideas from $D$-module theory require that one adds formal derivatives away from $K$-orbits. The final formula is:

\[
(\Delta_1^{+}, \lambda_1)|_k = \sum_{m \geq 0} \sum_i (-1)^i H^{top-i}(K/(B_1 \cap k), \mathcal{L}_{\lambda_1} \otimes S^m(n \cap p)).
\]

Here $b_1 = h_1 \oplus n$ with $Lie(B_1) = b_1$. (The lowest $K$-type is buried in the above formula for $m = 0$.)

To extract the lowest $K$-type one has to invert $\sum_{m \geq 0} S^m(n \cap p)$ to get

\[
\sum_{j=0}^{\dim(n \cap p)} (-1)^j \Lambda^j (n \cap p).
\]

Use Koszul Theorem here to set

\[
\sum_{j=0}^{\dim(n \cap p)} \sum_{m \geq 0} (-1)^j (\Lambda^j \otimes S^m)(n \cap p) = \mathbb{C}.
\]

Finally we obtained,

\[
H^{top}(K/(B_1 \cap K), \mathcal{L}_{\lambda_1}) = \sum_{j=0}^{\dim(n \cap p)} (-1)^j \text{standardrep} (\lambda_1 \otimes \Lambda^j (n \cap p))|_K.
\]
2. Formula for general Cartan subgroups

Let $H_i$ be a Cartan subgroup of $G$ and fix a set of positive imaginary roots $\Delta_i^{-\text{im}}$. We can extend $\Delta_i^{-\text{im}}$ to a positive root system $\Delta_i^+$ which is as $\theta$-stable as possible, that is if $\alpha$ is positive then either $\alpha$ is real, $\theta(\alpha) = -\alpha$ or $\alpha$ is not real and $\theta(\alpha)$ is positive. This gives some $\theta$-stable parabolic subalgebra $q_i = l_i \oplus u_i$ such that the roots in $l_i$ are all real and the ones in $u_i$ are not real. Let $b_i$ be the Borel subalgebra corresponding to $\Delta_i^+$. Let $Q$ be the unique conjugate of $Q_i$ containing $B$. Denote by $q = l \oplus u$ the parabolic subalgebra of $g$ associated to $Q$. Then the natural map $G/B \to G/Q$ carries $K.b_i \to K.q_i \simeq K/(Q_i \cap K)$, a closed orbit in $G/Q$. This says that the fiber over $q_i$ is $(L_i \cap K)/(H_i \cap K)$ which is open (orbit) in $L/(L \cap B)$.

The conclusion is that $K$-orbit of $b_i$ is a fiber bundle over closed $K/(Q_i \cap K)$. There is a standard representation of $L_\mathbb{R}$ related to the fiber $(L_i \cap K)/(H_i \cap K)$. It is the sections of the bundle defined by a character $\lambda_i|_{(H_i \cap K)}$:

$$\text{Ind}_{H_i \cap K}^L \lambda_i|_{(H_i \cap K)} = \bigoplus_{\tau: \text{irreducible rep of } L_i \cap K} \text{mult}(\lambda_i|_{(H_i \cap K)}) \in \tau|_{(H_i \cap K)}$$

2.1. Example. Let $G_s = GL(3, \mathbb{R})$ and $K_s = O(3, \mathbb{R})$ then $G = GL(3, \mathbb{C})$ and $K = O(3, \mathbb{C})$. $G/B$ is the variety of flags in $\mathbb{C}^3$.

The closed orbits of $K$ consist of flags $(L \subset L^\perp)$ where $L$ is a line of length zero. For example a line generated by $(1, 1, 0)$ included in the plane generated by $(1, i, 0)$ and $(0, 0, i)$. According to Witt’s theorem any two lines of length zero are conjugate by $K$.

The open orbits of $K$ consist of flags $(L \subset P)$ such that $L$ and $P^\perp$ are both of non zero length. An example would be the line generated by $(1, 0, 0)$ in plane $P$ generated by $(1, 0, 0)$ and $(0, 1, 0)$. A counter example would be the line generated by $(1, 0, 0)$ in plane $P$ generated by $(1, 0, 0)$ and $(0, 1, i)$ since in this case $P^\perp$ is generated by $(0, 1, i)$ which is of length zero. (Witt’s theorem says such pairs (line, plane) are a single orbit.

The stabilizer of $(1, 0, 0) \subset ((1, 0, 0), (0, 1, 0))$ is the set of upper triangular matrices in $K$ that is $(\pm 1, \pm 1, \pm 1)$.

The complete flag in $\mathbb{C}^3$ contains $O(3, \mathbb{C})/O(1, \mathbb{C})^3$ (open)

$\lambda$ is a character of $O(1, \mathbb{C})^3 = \{\epsilon_1, \epsilon_2, \epsilon_3\}$

Sections of this $\lambda$-bundle are

$$\text{Ind}_{O(1, \mathbb{C})^3}^O \lambda|_{(H_i \cap K)} = \text{sum of all } O(3, \mathbb{C}) \text{ irreducible representations } \tau \text{ with multiplicities } = \text{mult of } \lambda \text{ in } \tau|_{O(1, \mathbb{C})^3}$$

2.2. Zuckerman Formula. Trivial rep of $K = \sum_{\mathbb{K}GB-orbits:\mathcal{O}} (-1)^{\text{codim } \mathcal{O}} \text{strep}(\mathcal{O}, \text{triv}\lambda)|_K$

One of the terms in the sum is the standard representation for this open orbit restricted to $K$ + terms for lower dimensional orbits.

List of all orbits for $O(3, \mathbb{C})$ on complete flags:

- dim 3 open non-zero length $L \subset P$ with $P^\perp$ non-zero length
- dim 2 zero length $L \subset P \neq L^\perp$
- dim 2 non-zero length $L \subset P$ with $P^\perp$ zero length
Standard representations data includes $\lambda_i$, character of $H_i \cap K$. So we have this algebraic bundle over $K$ orbits and we are interested in sections. Since the $L_i$ orbit is open in the fiber one can differentiate in those directions (sections of bundle on $K$ orbit). We need to add derivatives that are transverse corresponding to $u_i \cap p$.

(This is away from $K$-orbit directions in which you can’t differentiate.)

**Theorem 2.1.** \( \text{StdReps}|_K = \sum_p (-1)^p H^{\text{top}} - p(K/(Q_i \cap K)), \text{stdrep for } L_i|_{L_i \cap K} \otimes S(u_i \cap p) \) with \( \text{top} = \dim_C K/(Q_i \cap K) \).

Hiding inside of the above formula is the lowest $K$-type

\[
H^{\text{top}}(K/(Q_i \cap K)), \text{lowest } (L_i \cap K) - \text{type of strep for } L_i|_{L_i \cap K} \otimes S(u_i \cap p).
\]

How do you write lowest $K$-type as combination of standard representations?

Use Zuckerman formula:

Lowest $L_i \cap K$-type = \( \sum_D \dim D \text{stdrep }|_{L_i \cap K} \)

where $D$ is an orbit of $L_i \cap K$ on $L_i/(L_i \cap B)$

Lowest $K$-type = \( \sum_D \dim D H^{\text{top}}(K/(Q_i \cap K)), \text{stdrep for } L_i|_{L_i \cap K}, D \)

We need to put the transverse derivatives in. Using Kozul identity

\[
S(u \cap p) \otimes \sum_{j=0}^{\dim(u \cap p)} (-1)^j \lambda^j(u \cap p) = \mathbb{C}.
\]

Lowest $K$-type =

\[
\sum_{j=0}^{\dim(u \cap p)} \sum_D (-1)^j H^{\text{top}}(K/(Q_i \cap K)), (\text{stdrep for } L_i|_{L_i \cap K}, D, \lambda_i) \otimes \lambda^j(u \cap p) \otimes S(u \cap p) = \sum_{\text{subset of roots of size } j \in u \cap p} \sum_D (-1)^{\text{codim } D} \text{stdrep for } G_q.
\]

To compute the Zuckerman terms one proceeds as follows:

1. For each Cartan subgroup $H_i$ construct a $\theta$-stable parabolic subalgebra $q_i = l_i \oplus u_i$ such that $L_i$ is split (all roots are real).

2. Call (cartan $(L_i)$) to obtain normalized involutions $\{\theta_i^j\}$. Zuckerman formula for $L_i$ is indexed by the KGB orbits for $L_i \cap K$ on $L_i/(B \cap L_i)$. These orbits are indexed by Cartan subgroups $\{\theta_i^j\}$ and correspond to $\prod_j W(L_i, H_i)/W(L_i \cap K, H_i^j)$.

What emerge are cosets $w.W(L_i, H_i)/W(L_i \cap K, H_i^j)$ corresponding to the characters $(w.\rho_{L_i} + \rho_{L_i})|_{(H_i^j \cap K)}$ contributing to the Zuckerman formula with some sign $(-1)^{\text{length(KGBELT)}}$.

3. For each $j$ set a pair $(m, \mu)$ with $m \in \mathbb{Z}$ and $\mu$ a character of $H_i^j \cap K$. 

4. Compute \[ \sum_{[j: \# \text{ of Cartans in } L_i]} \sum_{(m, \mu)} m \text{stdrep}(H_i^j, \lambda_i^j + \mu) \]

In general \( H_i^j \) will be more compact than \( H_i \) because of Cayley transform in real roots in \( L_i \). Roughly \( H_i \cap K \subseteq H_i^j \cap K \). So \( \lambda_i \to \lambda_i^j \). The roots of \( \alpha_1 \ldots \alpha_l \) in \( H_i^j \) are orthogonal to the real roots in \( L_i \) and \( H_i^j = (H_i \cap K).(SO_2)^l \). We define \( \chi_i^j = \chi_i \) and trivial on the \( SO_2 \) factors.

For the outer sum list all the roots of \( H_i^j \cap K \) in \( u \cap p \).

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