BRANCHING EXAMPLE: $Sp(4, \mathbb{R})$

DAVID A. VOGAN, JR.

1. Introduction

The purpose of these notes is to list the restrictions-to-K of continued standard representations of the split real group $Sp(4, \mathbb{R})$ as explained in "Branching to a maximal compact subgroup," published in *Harmonic Analysis, Group Representations, Automorphic Forms and Invariant Theory: In Honour of Roger E. Howe*, edited by Jian-Shu Li, Eng-Chye Tan, Nolan Wallach and Chen-Bo Zhu, Singapore University Press and World Scientific Publishing. Details of the parametrization are meant to follow the implementation in the atlas software sketched by Fokko du Cloux in 2006, partly implemented by Alfred Noel in 2007, and now (2008) being pursued by Marc van Leeuwen.

In particular this means that we will fix forever a complex reductive group G with Borel and Cartan subgroups $B \supset H$, defining a based root datum (notably including a character lattice X^* always identified with $\mathbb{Z}^\ell$ and a set of positive roots $\Delta^+ \subset X^*$). We fix also a "pinning" of G, which in the presence of the preceding choices means a collection of simple root vectors in the Lie algebra. We have in mind an inner class of real forms, and therefore an extended group $G^\Gamma = G \ltimes \mathbb{Z}/2\mathbb{Z}$. The non-trivial element of the second factor is called δ_0; its action on G preserves B and H, and permutes the simple root vectors in the pinning. Write

$$\theta_0 = \mathrm{Ad}(\delta_0) \in \mathrm{Aut}(H)$$

for the involutive automorphism defined by δ_0. This is an involutive automorphism of the based root datum, and defines the inner class of real forms. We will also fix an element

$$\delta_1 = t_1\delta_0 \qquad (t_1 \in H, \delta_1^2 \in Z(G)).$$

Notice that δ_1 also acts on H by θ_0. The atlas software does not actually compute a choice of t_1, but it could do so. What it *does* compute and remember is the grading of the imaginary roots induced by δ_1; this is called the "base grading."

All the other strong involutions considered by the software are of the form

$$\delta = \sigma\delta_1,$$

Date: October 1, 2008.

with σ in the Tits group (for H in G, with respect to the specified pinning). The Tits group is a subgroup of $N_G(H)$. Group-theoretically it is an extension of the Weyl group by

$$H(2) = \text{elts of order 2 in } H \simeq X_*/2X_*.$$

A little more explicitly,

$$1 \to H(2) \to \text{Tits group} \to W(G,H) \to 1.$$

The structure theory and the pinning provide a natural section of the last projection (*not* a group homomorphism) so that set-theoretically there is a natural identification

$$\text{Tits group} = H(2) \times W.$$

Recall that a *twisted involution* is an element $w \in W(G,H)$ with the property that the automorphism $\theta = w \circ \theta_0$ of H has order 2. A strong involution δ is stored by the software as a Tits group element $\sigma = (t,w)$; the actual strong involution is

$$\delta = (t,w)\delta_1.$$

In particular this means that w is a twisted involution in W; the action of δ on H is by the (Cartan) involution

$$\theta = w \circ \theta_0.$$

There is a structural subtlety here that I don't understand: *is* $(1,w)\delta_1$ *a strong involution?* (In fact we want it to have the same square in $Z(G)$ as δ_1.) If that isn't true, then more or less there needs to be a preferred torus part $t_w \in H(2)$ so that $(t_w,w)\delta_1$ is a strong involution (with the same square as δ_1). Then the set of all strong involutions related to δ (up to H conjugacy) becomes

$$(xt_w, w)\delta_1, \qquad x \in X_*^\theta/(1+\theta)X_*.$$

That is, the parameter x runs over some fiber group; it is recorded (in the class standardRepK) by a small bit vector; that is, by a bit vector of dimension at most the rank of G. (The product xt_w would fit in such a class as well, but it's pretty clear that is *not* what is wanted; all the elements x of the fiber group give strong involutions (although not necessarily attached to the same real form). This would not be true any more if we were storing a full element of $H(2)$.)

In the example below I'm going to assume that $(1,w)\delta_1$ is always a strong involution, so that $t_w = 1$ and we can record our strong involution just by the fiber group element

$$x \in X_*^\theta/(1+\theta)X_*.$$

Here is the root datum:

$$X^* = \mathbb{Z}^2 = X_*, \qquad \Delta^+ = \{(0,2),(1,-1),(1,1),(2,0)\}.$$

The first two positive roots listed are simple; call them

$$\alpha_1 = (0,2), \quad \alpha_2 = (1,-1).$$

The group W has order 8, acting by permutations and sign changes on $\mathbb{Z}^2$. The Tits group is generated by the torus part

$$H(2) = X_*/2X_* = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

and representatives for the simple reflections

$$\sigma_1 = \sigma_{\alpha_1}, \quad \sigma_2 = \sigma_{\alpha_2}$$

satisfying

$$\sigma_1^2 = (0,1) \in H(2), \qquad \sigma_2^2 = (1,1) \in H(2).$$

Since I am writing about so many small integers, I found it confusing to write elements of $\mathbb{Z}/2\mathbb{Z}$ as 0 and 1. I will therefore try to write them as e and o (for "even" and "odd") instead. The relations above become

$$\sigma_1^2 = (eo) \in H(2), \qquad \sigma_2^2 = (oo) \in H(2).$$

The basic strong involution δ_1 satisfies

- δ_1 acts trivially on X^* (all roots are imaginary)
- $\delta_1^2 = (oo) \in H(2)$ (the element "minus identity" in $Z(G)$)
- α_1 and α_2 are both noncompact.

We work always with $\widetilde{H}$, the "ρ double cover of H": the double cover defined by the square root of the algebraic character 2ρ, the sum of the positive roots. In our example $2\rho = (4,2)$, which has the square root $(2,1)$ already as a character of H. It follows that the ρ double cover is trivial in this example, and I will ignore it.

On each Cartan subgroup, we work with algebraic characters λ of $\widetilde{H}^\theta$. These are parametrized by

$$(X^* + \rho)/(1-\theta)X^*;$$

in our case, by $X^*/(1-\theta)X^*$. For each θ (representing a conjugacy class of distinguished involutions), each x in the corresponding fiber group, and each $\lambda \in (X^* + \rho)/(1-\theta)X^*$, we therefore get a "continued standard representation restricted to K" $\pi(x,\lambda)$. This is the restriction to K of a virtual Harish-Chandra module for the real form corresponding to x (that is, to the strong involution $(xt_w, w)\delta_1$. It is therefore an (infinite) sum of irreducible representations of K, occurring with (finite) integer multiplicities that we wish to compute.

In order for this virtual representation to be an honest representation (with non-negative multiplicities of representations of K), it is enough for the parameter to be "standard." This means that λ is weakly dominant with respect to the positive imaginary roots (for θ):

$$\langle \lambda, \gamma^\vee \rangle \geq 0, \qquad (\gamma \in \Delta^+, \theta(\gamma) = \gamma).$$

This condition should be tested in the software by "IsStandard."

Some of the standard representations are zero: assuming that the parameter is standard, the condition to vanish is precisely

$$\langle \lambda, \gamma \rangle = 0, \qquad \text{some simple imaginary } \gamma \text{ which is compact.}$$

Of course this condition depends on the grading of the imaginary roots, and therefore on x.

The Hecht-Schmid character identities sometimes allow one to write a standardRepK as a sum of one or two standardRepK's on a (more compact) Cayley transformed Cartan subgroup. We are interested ultimately only in standard representations admitting no such expression. These are called "final"; the condition for a standard representation to be final is

$$\lambda(m_\beta) = -1, \qquad \text{all simple real roots } \beta.$$

Here $m_\beta \in H(2)$ is the image of the coroot for β. Of course the condition can be written

$$\langle \lambda, \beta^\vee \rangle \quad \text{is odd, all simple real roots } \beta.$$

This is the condition that the software should test with "IsFinal." (The corresponding "finalization" function should start with a simple real root β so that $\langle \lambda, \beta^\vee \rangle$ is even, and rewrite the standard representation using a Cayley transform through the real root β.)

2. Compact Cartan subgroup

The involution is $\theta_0 = \mathrm{Ad}(\delta_1)$, which is trivial on H. There are four (conjugacy classes of) strong involution

$$(x_{\epsilon_1,\epsilon_2}, 1)\delta_1, \qquad \epsilon_i \in \mathbb{Z}/2\mathbb{Z}.$$

Here are the corresponding gradings of the (simple) imaginary roots:

	α_1	α_2
$x_{ee}\delta_1$	n	n
$x_{eo}\delta_1$	n	c
$x_{oe}\delta_1$	n	c
$x_{oo}\delta_1$	n	n

Here are the four two-parameter families of standardRepK's (the limits of discrete series).

$\pi_0(x_{ee}, (l_1, l_2))$	$l_1 \geq l_2 \geq 0$	
$\pi_0(x_{eo}, (l_1, l_2))$	$l_1 \geq l_2 \geq 0$	zero if $l_1 = l_2$
$\pi_0(x_{oe}, (l_1, l_2))$	$l_1 \geq l_2 \geq 0$	zero if $l_1 = l_2$
$\pi_0(x_{oo}, (l_1, l_2))$	$l_1 \geq l_2 \geq 0$	

Notice that everything is final, since there are no real roots.

3. Long real root Cartan subgroup

We next consider the Cartan subgroup corresponding to the strong involution $\sigma_1\delta_1$: Cayley transform through the long simple root. The Cartan involution acts by

$$\theta_1(l_1,l_2) = (l_1,-l_2).$$

Therefore the imaginary roots are $\pm(2,0)$, and the real roots are $\pm(0,2)$. The fiber group turns out to have order 2, represented by a parity on the first coordinate; these elements are written x_{ϵ_1}, with $\epsilon_1 \in \mathbb{Z}/2\mathbb{Z}$. The imaginary roots are always noncompact. The lattice $(1-\theta_1)X^*$ is $\{(0,2l) \mid l \in \mathbb{Z}\}$, so the parameters λ are taken from $X^*/(1-\theta_1)X^* = \mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Here are the families of standardRepK's.

$\pi_1(x_0,(l_1,\epsilon_2))$	$l_1 \geq 0, \epsilon_2 \in \mathbb{Z}/2\mathbb{Z}$	final if and only if $\epsilon_2 = o$
$\pi_1(x_1,(l_1,\epsilon_2))$	$l_1 \geq 0, \epsilon_2 \in \mathbb{Z}/2\mathbb{Z}$	final if and only if $\epsilon_2 = o$

4. Short real root Cartan subgroup

We next consider the Cartan subgroup corresponding to the strong involution $\sigma_2\delta_1$: Cayley transform through the short simple root. The Cartan involution acts by

$$\theta_2(l_1,l_2) = (l_2,l_1).$$

Therefore the imaginary roots are $\pm(1,1)$, and the real roots are $\pm(1,-1)$. The fiber group turns out to be trivial; the identity element will be written $x_{..}$. The imaginary roots are always noncompact. The lattice $(1-\theta_2)X^*$ is $\{(l,-l) \mid l \in \mathbb{Z}\}$, so the parameters λ are taken from $X^*/(1-\theta_2)X^* \simeq \mathbb{Z}$; the isomorphism from left to right adds up the two coordinates. Here are the families of standardRepK's.

$\pi_2(x_{..},l)$	$l \geq 0$	final if and only if l is odd.

5. Split Cartan subgroup

We next consider the Cartan subgroup obtained by two successive Cayley transforms, for example $\sigma_{(2,0)}\sigma_{(0,2)}\delta_1$. The Cartan involution acts by

$$\theta_2(l_1,l_2) = -(l_1,l_2).$$

The fiber group is trivial; we write the identity element as x. The lattice $(1-\theta_3)X^*$ is $2X^*$, so the parameters λ are taken from $X^*/2X^* = (\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})$. The four standardRepK's are $\pi_3(x,(\epsilon_1,\epsilon_2))$, with $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$. The condition to be final for the long simple root α_1 is that $\epsilon_2 = o$; for the short simple root α_2 it is $\epsilon_1 - \epsilon_2 = o$. There is therefore just one final parameter:

$$\pi_3(x,eo).$$

6. Hecht-Schmid character identities

In this section we record the Hecht-Schmid character identities. We will use these (first) to relate non-standard continued standard representations to (more) standard representations on (weakly) more split Cartan subgroups; and (second) to relate non-final standard representations to (more) final standard representations on (strictly) more compact Cartan subgroups.

The first identitities relate the compact Cartan with trivial involution θ_0 to the long real root Cartan with involution θ_1 (reflection in the long simple root):

$$\begin{aligned} \pi_0(x_{ee},(l_1,l_2)) + \pi_0(x_{eo},(l_1,-l_2)) &= \pi_1(x_e,(l_1,(l_2 \bmod 2\mathbb{Z}))). \\ \pi_0(x_{oe},(l_1,l_2)) + \pi_0(x_{oo},(l_1,-l_2)) &= \pi_1(x_o,(l_1,(l_2 \bmod 2\mathbb{Z}))). \end{aligned} \tag{6.1}$$

The next identity relates the compact Cartan with the short real root Cartan and θ_2 by Cayley transform in the short simple root:

$$\pi_0(x_{ee},(l_1,l_2)) + \pi_0(x_{oo},(l_2,l_1)) = \pi_2(x_{\cdot\cdot}, l_1+l_2). \tag{6.2}$$

The last identity is only defined when the short simple root is noncompact; that is why the fiber elements x_{eo} and x_{oe} are not present on the left. Those elements contribute to the "easy" Hecht-Schmid identities

$$\begin{aligned} \pi_0(x_{eo},(l_1,l_2)) + \pi_0(x_{eo},(l_2,l_1)) &= 0 \\ \pi_0(x_{oe},(l_1,l_2)) + \pi_0(x_{oe},(l_2,l_1)) &= 0. \end{aligned} \tag{6.3}$$

Next is the Cayley transform through $2e_1$ relating θ_1 and θ_3:

$$\pi_1(x_e,(l_1,\epsilon_2)) + \pi_1(x_o,(-l_1,\epsilon_2)) = \pi_3(x,(\epsilon_2, l_1 \bmod 2\mathbb{Z})). \tag{6.4}$$

Last is the Cayley transform through e_1+e_2 relating θ_2 and θ_3. This is the only "type II" Cayley transform for G, having two terms on the right:
(6.5)

$$\pi_2(x_{\cdot\cdot}, l) + \pi_2(x_{\cdot\cdot}, -l) = \pi_3(x,(e,(l \bmod 2\mathbb{Z})) + \pi_3(x,(o, o + (l \bmod 2\mathbb{Z}))).$$

7. Explicit lowest K-types

A fundamental part of the algorithm is the fact that each (standard final normal limit) representation has a unique lowest K-type, and that passage to lowest K-types makes a bijection between these standard representations and $\widehat{K}$. In order to implement the algorithm it is enough to take this bijection as parametrizing $\widehat{K}$, but users will probably want to relate it to more traditional parametrizations, such as by highest weight. The problem of translation is briefly discussed in the paper (sections 11 and 13). Here I will just record the answer for this example.

The (complex) group K is $GL(2,\mathbb{C})$, the complexification of the unitary group $U(2)$. Its based root datum lives on $\mathbb{Z}^2$, with one positive root e_1-e_2. Irreducible representations of K are parametrized by dominant weights:

$$\widehat{K} = \{\mu(m_1,m_2) \mid m_1 \geq m_2\}.$$

(It would be more "natural" to realize the root datum of K inside that of G, as the compact roots attached to $(x_{ee},1)\delta_1$; again this would live on $\mathbb{Z}^2$, but the positive root would be e_1+e_2.)

Here are the lowest K-types of the final standard representations.

standard reps	conditions	lowest K-types	explicit list
$\pi_3(x,eo)$		$\mu(0,0)$	
$\pi_2(x_{..},l)$	$l\geq 0$ odd	$\mu(\frac{l+1}{2},\frac{-l-1}{2})$	$\mu(1,-1),\ \mu(2,-2)\cdots$
$\pi_1(x_e,(l_1,o))$	$l_1\geq 0$	$\mu(l_1+1,1)$	$\mu(1,1),\ \mu(2,1),\ \mu(3,1)\cdots$
$\pi_1(x_o,(l_1,o))$	$l_1\geq 0$	$\mu(-1,-(l_1+1))$	$\mu(-1,-1),\ \mu(-1,-2)\cdots$
$\pi_0(x_{ee},(l_1,l_2))$	$l_1\geq l_2\geq 0$	$\mu(l_1+1,-l_2)$	
$\pi_0(x_{eo},(l_1,l_2))$	$l_1>l_2\geq 0$	$\mu(l_1+1,l_2+2)$	
$\pi_0(x_{oe},(l_1,l_2))$	$l_1>l_2\geq 0$	$\mu(-l_2-2,-l_1-1)$	
$\pi_0(x_{oo},(l_1,l_2))$	$l_1\geq l_2\geq 0$	$\mu(l_2,-l_1-1)$	

8. Formulas for irreducibles of K in terms of (continued) standards

In this section we record the (finite) formulas for irreducible representations of K as alternating sums of (continued) standard representations of G. Whenever $\pi_m(x,\lambda)$ is a final standard limit representation, write $\mu_m(x,\lambda)$ for its lowest K-type. (Often it's possible to attach a meaning to this symbol even if (x,λ) is not standard: some finite-dimensional virtual representation of K, not necessarily irreducible or non-zero. More about this later, perhaps.)

We begin with the compact Cartan, the trivial involution θ_0, and the strong involution $(x_{ee},1)\delta_1$. The noncompact positive imaginary roots are

$$\{(0,2),(1,-1),(2,0)\}.$$

The formula we want is a sum over subsets of this three-element set:

(8.1)

$$\begin{aligned}\mu_0(x_{ee},(l_1,l_2)) = {} & \pi_0(x_{ee},(l_1,l_2)) - \boldsymbol{\pi_0(x_{ee},(l_1,l_2+2))} - \pi_0(x_{ee},(l_1+2,l_2)) \\ & -\boldsymbol{\pi_0(x_{ee},(l_1+1,l_2-1))} + \pi_0(x_{ee},(l_1+2,l_2+2)) \\ & +\pi_0(x_{ee},(l_1+1,l_2+1)) + \boldsymbol{\pi_0(x_{ee},(l_1+3,l_2-1))} \\ & -\pi_0(x_{ee},(l_1+3,l_2+1)).\end{aligned}$$

There are three terms on the right that may fail to be standard (even if $l_1\geq l_2\geq 0$): they are the boldfaced terms. The Hecht-Schmid character identities can be used to express them in terms of standard representations. For example,

$$\pi_0(x_{ee},(l_1,l_2+2)) = -\pi_0(x_{oo},(l_2+2,l_1)) + \pi_2(x_{..},(l_1+l_2+2));$$

if $l_1\geq l_2\geq 0$ but $l_2+2>l_1$, then the two terms on the right are standard.

The formula for x_{oo} is identical, since the grading of the imaginary roots defined by it is the same. For x_{oe}, the noncompact roots are

$$\{(0,2),(1,1),(2,0)\}.$$

The formula we want is therefore

(8.2)

$$\begin{aligned}\mu_0(x_{oe},(l_1,l_2)) = \pi_0(x_{oe},(l_1,l_2)) - \pi_{\mathbf{0}}(\mathbf{x_{oe}},(\mathbf{l_1},\mathbf{l_2}+\mathbf{2})) - \pi_0(x_{oe},(l_1+2,l_2))\\ -\pi_0(x_{oe},(l_1+1,l_2+1)) + \pi_0(x_{oe},(l_1+2,l_2+2))\\ +\pi_0(x_{oe},(l_1+3,l_2+1)) + \pi_{\mathbf{0}}(\mathbf{x_{oe}},(\mathbf{l_1}+\mathbf{1},\mathbf{l_2}+\mathbf{3}))\\ -\pi_0(x_{oe},(l_1+3,l_2+3)).\end{aligned}$$

The formula for x_{eo} is identical.

At the other extreme is the formula for $\mu_3(x, oe)$ (the trivial representation of K) attached to the split Cartan subgroup, and involving a sum over `kgb` for G (which is the Levi subgroup generated by real roots). Always the parameter λ is represented by something in the W orbit of ρ. Here is the formula.

(8.3)

$$\begin{aligned}\mu_3(x,eo) = \pi_3(x,eo) - \pi_{\mathbf{1}}(\mathbf{x_e},(\mathbf{1},\mathbf{e})) - \pi_{\mathbf{1}}(\mathbf{x_o},(\mathbf{1},\mathbf{e}))\\ +\pi_1(x_e,(2,o)) + \pi_1(x_o,(2,o)) - \pi_2(x_{..},1) + \pi_2(x_{..},3)\\ -\pi_0(x_{ee},(2,1)) - \pi_0(x_{oe},(2,1)) - \pi_0(x_{eo},(2,1)) - \pi_0(x_{oo},(2,1)).\end{aligned}$$

All the terms are standard, but the two boldfaced terms fail to be final.

9. Cheating to get formulas for irreducibles of K

Suppose G is any reductive algebraic group, G^d is the (semisimple) derived group, and $\widetilde{G^d}$ is the universal cover of G^d. Any real form of G defines a real form of $\widetilde{G^d}$. A representation of $G(\mathbb{R})$ is unitary if and only if it is Hermitian, and some constituent of its restriction/pullback to $\widetilde{G^d}(\mathbb{R})$ is unitary. (I may not have made that statement correctly, but something equally powerful is true.) So to study unitary representations, there is no loss of generality in restricting to simply connected semisimple G. (This is *not* true for the problem of branching to K; I'm only saying that for applications of branching to unitary representations, you can take G simply connected.)

If G is semisimple and simply connected, then K is connected, so irreducible representations of K are parametrized by certain characters of the (connected) torus $T = H^{\theta_0}$. Here is how to write these irreducible representations in terms of continued standard representations.

Theorem 9.1. *Suppose that $(x,1)\delta_1$ is a strong involution representing the Cartan involution θ_0, and that $K = G^{\theta_0}$ is connected. (This is automatic if for example G is semisimple and simply connected.) Write $T = H^{\theta_0}$, a maximal torus in K.*

- *The character lattice $X^*(T)$ may be identified with the (torsion-free) quotient $X^*(H)/X^*(H)^{-\theta_0}$. (Because T is connected, $X^*(H)^{-\theta_0} = (1-\theta_0)X^*(H)$.) The cocharacter lattice $X_*(T)$ may be identified with $X_*(H)^{\theta_0}$.*
- *The positive roots of T in K are the restrictions to T of the positive compact imaginary roots β of H in G, together with the restrictions to T of one root from each pair $\{\alpha, \theta_0(\alpha)\}$ of complex positive roots. The corresponding coroots are $\beta^\vee$ and $\alpha^\vee + \theta_0(\alpha^\vee)$*
- *Irreducible representations of K correspond to K-dominant weights $\mu \in X^*(T)$. These are the restrictions of weights $\tilde{\mu} \in X^*(H)$ satisfying*
$$\langle \tilde{\mu}, \beta^\vee \rangle \geq 0 \qquad (\beta \textit{ compact positive imaginary}),$$
$$\langle \tilde{\mu}, \alpha^\vee + \theta_0(\alpha^\vee) \rangle \geq 0, \qquad (\alpha \textit{ complex positive}).$$
- *The set of positive weights of T on the -1 eigenspace of θ_0 is the set of restrictions to T of*
$$S = \{\textit{pos noncpt imag roots}\} \cup \{\textit{root from each cplx pos pr } \{\alpha, \theta_0(\alpha)\}\}.$$
- *Suppose μ is a dominant weight for K. Write $2\rho_c$ for the sum of the positive roots of T in K. Finally define*
$$\lambda(\mu) = \mu + 2\rho_c - \rho \in (X^* + \rho)/(1-\theta_0)X^*.$$
Then the continued standard representation $\pi_0(x, \lambda)$ has "lowest K-type" of highest weight μ, in the precise sense that
$$\textit{irr of hwt } \mu = \sum_{A \subset S} (-1)^{|A|} \pi_0(x, \lambda(\mu) + 2\rho(A)).$$

The weight $\lambda(\mu)$ may be very far from dominant for the imaginary roots, so the terms on the right in the last formula are very far from standard. Nevertheless they can be rewritten in terms of standard representations using the Hecht-Schmid character identities, and this may be easier than implementing the Zuckerman character identity (involving the sum over `kgb` for some Levi L generated by real roots).

About general G: the K-dominant characters of $T = H^{\theta_0}$ parametrize irreducible representations of the group $K^\sharp$ (described in the paper) which is between K_0 and K. The sum of continued standard representations in the theorem still makes sense. I think that the left side of the formula becomes
$$\mathrm{Ind}_{K^\sharp}^{K}(\text{irreducible of } K^\sharp).$$

10. Cheating formulas for $Sp(4,\mathbb{R})$

In this section we will make explicit the formulas from the last theorem in our example, and see what is required to put standard representations on the right. If we use the (trivial) fiber group element x_{ee}, then the only compact imaginary root is $e_1 + e_2$, so
$$\widehat{K} = \{\mu_{ee}(m_1, m_2) \mid m_1 + m_2 \geq 0\}.$$

The subscript on μ reflects the dependence of the parametrization on the choice of strong involution. Using the three other fiber group elements would lead to three other parametrizations, related to each other and to the parametrization of section 7 by

$$\begin{aligned}\mu_{oe}(m_1, m_2) &= \mu_{ee}(m_1, -m_2) = \mu(m_1, m_2),\\ \mu_{eo}(m_1, m_2) &= \mu_{ee}(-m_2, m_1) = \mu(-m_2, -m_1),\\ \mu_{oo}(m_1, m_2) &= \mu_{ee}(m_2, m_1) = \mu(m_2, -m_1).\end{aligned} \tag{10.1}$$

Here is a formula for any irreducible of K in terms of continued standard representations:

(10.2)

$$\begin{aligned}\mu_{ee}(m_1, m_2) = \pi_0(x_{ee}, (m_1 - 1, m_2)) &- \pi_0(x_{ee}, (m_1 - 1, m_2 + 2))\\ -\pi_0(x_{ee}, (m_1 + 1, m_2)) &- \pi_0(x_{ee}, (m_1, m_2 - 1))\\ +\pi_0(x_{ee}, (m_1 + 1, m_2 + 2)) &+ \pi_0(x_{ee}, (m_1, m_2 + 1))\\ +\pi_0(x_{ee}, (m_1 + 2, m_2 - 1)) &- \pi_0(x_{ee}, (m_1 + 2, m_2 + 1)).\end{aligned}$$

This formula is obtained from equation (8.1) by substituting for (l_1, l_2) the parameter

$$(m_1, m_2) + 2\rho_c - \rho = (m_1, m_2) + (1, 1) - (2, 1) = (m_1 - 1, m_2)$$

from Theorem 9.1. The requirement now is only that $m_1 + m_2 \geq 0$, so any of the terms on the right can fail to be standard. Here is the formula for the trivial representation of K:

$$\begin{aligned}\mu_{ee}(0, 0) = \pi_0(x_{ee}, (-1, 0)) &- \pi_0(x_{ee}, (-1, 2))\\ -\pi_0(x_{ee}, (1, 0)) &- \pi_0(x_{ee}, (0, -1))\\ +\pi_0(x_{ee}, (1, 2)) &+ \pi_0(x_{ee}, (0, 1))\\ +\pi_0(x_{ee}, (2, -1)) &- \pi_0(x_{ee}, (2, 1)).\end{aligned} \tag{10.3}$$

Only the third and last terms are standard. The next to last is standardized by the Hecht-Schmid identity

$$\pi_0(x_{ee}, (2, -1)) = -\pi_0(x_{eo}, (2, 1)) + \pi_1(x_e, (2, o)). \tag{10.4}$$

For the two preceding terms of (10.3), we need the Hecht-Schmid identity (6.2):

$$\pi_0(x_{ee}, (0, 1)) = -\pi_0(x_{oo}, (1, 0)) + \pi_2(x_{..}, 1), \tag{10.5}$$

$$\pi_0(x_{ee}, (1, 2)) = -\pi_0(x_{oo}, (2, 1)) + \pi_2(x_{..}, 3). \tag{10.6}$$

Applying the Hecht-Schmid identity (6.1) to the fourth term of (10.3) gives

$$\pi_0(x_{ee}, (0, -1)) = -\pi_0(x_{eo}, (0, 1)) + \pi_1(x_e, (0, o)).$$

The second term on the right is standard, but to the first we need to apply the "easy" Hecht-Schmid identity (6.3)

$$\pi_0(x_{eo}, (0, 1)) = -\pi_0(x_{eo}, (1, 0)).$$

This gives finally a formula in terms of standard representations

$$\pi_0(x_{ee},(0,-1)) = +\pi_0(x_{eo},(1,0)) + \pi_1(x_e,(0,o)). \tag{10.7}$$

Applying the short simple root identity to the first and second terms of (10.3) gives

$$\begin{aligned} \pi_0(x_{ee},(-1,0)) &= -\pi_0(x_{oo},(0,-1)) + \pi_2(x_{\cdot\cdot},-1),\\ \pi_0(x_{ee},(-1,2)) &= -\pi_0(x_{oo},(2,-1)) + \pi_2(x_{\cdot\cdot},1). \end{aligned} \tag{10.8}$$

Almost all the terms on the right need further work. The Hecht-Schmid identity (6.1) says

$$\pi_0(x_{oo},(0,-1)) = -\pi_0(x_{oe},(0,1)) + \pi_1(x_o,(0,o)),$$

$$\pi_0(x_{oo},(2,-1)) = -\pi_0(x_{oe},(2,1)) + \pi_1(x_o,(2,o)). \tag{10.9}$$

Now everything on the right is standard except the first term on the right in the first equation. For that we need the easy identity (6.3)

$$\pi_0(x_{oe},(0,1)) = -\pi_0(x_{oe},(1,0)),$$

leading to the formula

$$\pi_0(x_{oo},(0,-1)) = \pi_0(x_{oe},(1,0)) + \pi_1(x_o,(0,o)). \tag{10.10}$$

To fix the nonstandard term π_2, we use the Hecht-Schmid identity (6.5):

$$\pi_2(x_{\cdot\cdot},-1) = -\pi_2(x_{\cdot\cdot},1) + \pi_3(x,eo) + \pi_3(x,oe)$$

The two terms on the right are standard, but the second is not final. It can be made final using the Hecht-Schmid identity (6.4)

$$\pi_3(x,oe) = \pi_1(x_e,(0,o)) + \pi_1(x_o,(0,o)).$$

Putting these together gives

$$\pi_2(x_{\cdot\cdot},-1) = -\pi_2(x_{\cdot\cdot},1) + \pi_1(x_e,(0,o)) + \pi_1(x_o,(0,o)) + \pi_3(x,oe). \tag{10.11}$$

Now we can plug formulas (10.9), (10.10), and (10.11) into (10.8), obtaining (after canceling a couple of terms)

(10.12)
$$\begin{aligned} \pi_0(x_{ee},(-1,0)) &= -\pi_0(x_{oe},(1,0)) + \pi_1(x_e,(0,o)) - \pi_2(x_{\cdot\cdot},1) + \pi_3(x,oe)\\ \pi_0(x_{ee},(-1,2)) &= \pi_0(x_{oe},(2,1)) - \pi_1(x_o,(2,o)) + \pi_2(x_{\cdot\cdot},1). \end{aligned}$$

Having rewritten all the terms in (10.3), we can now plug (10.4), (10.5), (10.7), and (10.12) into (10.3), obtaining

$$\begin{aligned}\mu_{ee}(0,0) = \; & -\pi_0(x_{oe},(1,0)) + \pi_1(x_e,(0,o)) - \pi_2(x_{\cdot\cdot},1) + \pi_3(x,oe)\\ & -\pi_0(x_{oe},(2,1)) + \pi_1(x_o,(2,o)) - \pi_2(x_{\cdot\cdot},1)\\ & -\pi_0(x_{ee},(1,0))\\ & -\pi_0(x_{eo},(1,0)) - \pi_1(x_e,(0,o))\\ & -\pi_0(x_{oo},(2,1)) + \pi_2(x_{\cdot\cdot},3)\\ & -\pi_0(x_{oo},(1,0)) + \pi_2(x_{\cdot\cdot},1)\\ & -\pi_0(x_{eo},(2,1)) + \pi_1(x_e,(2,o))\\ & -\pi_0(x_{ee},(2,1)).\end{aligned}$$

Here we have made one line for each term on the right in (10.3). Making two cancellations and rearranging the terms by Cartan, we are left with

(10.13)

$$\begin{aligned}\mu_{ee}(0,0) = \pi_3(x,oe) - \pi_2(x_{\cdot\cdot},1) + \pi_2(x_{\cdot\cdot},3) + \pi_1(x_e,(2,o)) + \pi_1(x_o,(2,o))\\ -\pi_0(x_{ee},(1,0)) - \pi_0(x_{oe},(1,0)) - \pi_0(x_{eo},(1,0)) - \pi_0(x_{oo},(1,0))\\ -\pi_0(x_{ee},(2,1)) - \pi_0(x_{oe},(2,1)) - \pi_0(x_{eo},(2,1)) - \pi_0(x_{oo},(2,1)).\end{aligned}$$

This is essentially Zuckerman's formula (8.3), after the two non-final terms π_1 there are written as sums of limits of discrete series π_0.