Improved recursion formulas for KLV polynomials

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1 Introduction

The algorithm described in [2] for computing KLV polynomials, now implemented in `atlas`, has a mathematically unsatisfactory character. We have a partially ordered finite set X of parameters (for irreducible or standard representations) and we wish to compute a set of polynomials

$$\{P_{x,y} \in \mathbb{Z}[q] \mid x \in X, y \in X\}. \tag{1.1}$$

The set X has a length function

$$\ell\colon X \to \mathbb{N} \tag{1.2}$$

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which is compatible with the partial order in the sense that

$$x < y \Rightarrow \ell(x) < \ell(y).$$

Some of the fundamental properties of the polynomials (very closely connected to the geometric interpretation given to them by Kazhdan and Lusztig) include

$$P_{y,y} = 1, \qquad P_{x,y} \neq 0 \Rightarrow x \leq y, \tag{1.3}$$

and most important of all

$$\deg P_{x,y} \leq (\ell(y) - \ell(x) - 1)/2 \qquad (x < y). \tag{1.4}$$

The set X has additional structure related to the set S of simple reflections. Attached to each $x \in X$ is a *descent set for* x

$$\tau(x) \subset S; \tag{1.5}$$

the complement of $\tau(x)$ is the *ascent set for* x. Each descent has a type, which is either compact imaginary, complex, real type I, or real type II (`ic, C-, r1, r2`). Similarly, each ascent has a type which is real non-parity, complex, noncompact imaginary type II, or noncompact imaginary type I (`rn, C+, i2, i1`). Attached to each descent s of x is a set of 0, 1, 2, or 1 covers in the partial order (with the smaller elements, which are of length one less, written to the left):

$$\begin{aligned} \{\} &\longleftarrow x \qquad (\text{type } \texttt{ic})\\ \{x'\} &\longleftarrow x \qquad (\text{type } \texttt{C-})\\ \{x', x''\} &\longleftarrow x \qquad (\text{type } \texttt{r1})\\ \{x'\} &\longleftarrow x \qquad (\text{type } \texttt{r2}). \end{aligned} \tag{1.6}$$

We call the (zero, one or two) elements $\{x', x''\}$ the *s-descents of x*. Similarly, attached to each ascent s there are 0, 1, 2, or 1 elements of X covering x:

$$\begin{aligned} x &\longleftarrow \{\} \qquad (\text{type } \texttt{rn})\\ x &\longleftarrow \{x'\} \qquad (\text{type } \texttt{C+})\\ x &\longleftarrow \{x', x''\} \qquad (\text{type } \texttt{i2})\\ x &\longleftarrow \{x'\} \qquad (\text{type } \texttt{i1}). \end{aligned} \tag{1.7}$$

The elements $\{x', x''\}$ are called the *s-ascents of x*.

These special covers "generate" the Bruhat order on X in a simple way which is worth recalling.

Proposition 1.8. *Suppose $y \in X$.*

1. *If y has a complex s-descent y', then every element x covered by y is either equal to y', or is an s-ascent of some element covered by y'.*
2. *If y has a real type 1 s-descent $\{y', y''\}$, then every element x covered by y is either equal to y' or y'', or is an s-ascent of some element covered by y'.*
3. *Suppose y has no complex descents. Then every element covered by y is an s-descent of y (by some real root of type I or type II).*

In particular, if every descent of y is compact imaginary, then y is minimal in X.

We do not need the second fact (about real type I descents) to describe the Bruhat order, but include it for completeness.

Here is the approximate nature of the present algorithm for computing the polynomials $P_{x,y}$. There is first of all an induction on $\ell(y)$, and then, for the collection of all y of a certain length, a *downward* induction on $\ell(x)$. We seek a descent s for y that is either complex or real type I; this is called a *direct recursion.* We then fix (one of the two, in the real case) s-descent[s] y' for y. The point is that the ascent for y' is either complex or type I noncompact imaginary, so *the s-ascent of y' is precisely $\{y\}$.* We get a recursion formula for $P_{x,y}$: this is a main term involving (one or two) $P_{*,y'}$, minus a "μ-correction" which involves various $P_{*,z}$ with $z < y'$. (These formulas are more or less on page 249 of [2] (where there are a few typos), or top of page 8 in [1], or [3], Proposition 6.14, Case I.)

This leaves the case when there is no direct recursion for y; that is, that every descent of y is either compact or type II real. For every type II real descent s, leading to a single y' of length one less, the s-ascent of y' is two elements $\{y, s \times y\}$. We get computable "recursion formulas" as above for sums $P_{x,y} + P_{x,s\times y}$. The "thicket" of y consists of all the elements z that can be reached by successive applications of these real cross actions (by various simple s). All elements of the thicket have the same length as y. There is a lemma ([2], Lemma 6.2, or [3], Lemma 6.7) that for any $x < y$, we can find a z in the thicket so that s is a descent for z, and s is a non-real ascent for x. In this case (letting $\{x'\}$ or $\{x', x''\}$ be the s-ascent[s] of x) there is an easy recursion formula

$$P_{x,z} = P_{x',z} \quad \text{or} \quad P_{x,z} = P_{x'z} + P_{x'',z} \tag{1.9}$$

([2], page 250, or top of page 5 in [1], or [3], Proposition 6.14, Case II). Then this can be plugged into the various formulas for $P_{x,z_1} + P_{x,z_2}$ in the thicket, finally computing $P_{x,y}$.

What we will explain here is a modification of the algorithm which works on Bruhat intervals and avoids thickets. First we'll say a bit more about the Bruhat order on X.

2 Bruhat order

Definition 2.1. Suppose $x \in X$ and $s \in S$. The *s-upward smear $s \sim x$ of x* is a subset of X consisting of x and zero, one, or two additional elements:

1. If s is a descent for x (`ic, r1, r2, C-`), or s is real nonparity (`rn`) then $s \sim x = \{x\}$.
2. If s is a complex ascent or type I imaginary (`C+, i1`) with s-ascent x', then $s \sim x = \{x, x'\}$.
3. If s is type II imaginary (`i2`) with s-ascent $\{x', x''\}$, then $s \sim x = \{x, x', x''\}$.

If $Z \subset X$, then we define the *s-upward smear $s \sim Z$ of Z* to be

$$s \sim Z = \cup_{z \in Z} s \sim z.$$

The one surprising feature of this definition is that if s is type II real for x, we *do not* include the real cross action $s \times x$ in the s-upward smear.

Proposition 2.2. *Suppose $y \in X$. Write*

$$X^{\leq y} = \{z \in X \mid z \leq y\}$$

for the Bruhat interval.

1. *If y is the unique s-ascent of y' (that is, if s is* `C-` *or* `r1` *for y) then*

$$X^{\leq y} = s \sim X^{\leq y'}.$$

2. *If is s is type II real for y, so that the $\{y, s \times y\}$ is the s-ascent of y', then*

$$X^{\leq y} \cup X^{\leq s \times y} = s \sim X^{\leq y'}.$$

3. *If y has no complex descents, then*

$$X^{\leq y} = \{y\} \bigcup_{\substack{y' \text{ real} \\ \text{descent of } y}} X^{\leq y'}.$$

Sketch of proof. Parts 1)–2) are based on the definition of the Bruhat order as the transitive closure of the relation of nonvanishing KL polynomial. In each of these cases we have a formula for $P_{x,y}$ (or $P_{x,y} + P_{x,s\times y}$) which has a main term that's a nontrivial combination of $P_{x',y'}$ (with $x \in s \sim x'$) minus a correction term involving $P_{x'',y''}$ with $y'' < y'$. The claims follow. Part 3) is just a restatement of the last part of Proposition 1.8 in the introduction. □

In the third case of the proposition, the Bruhat interval below y is contained in the subset of X generated by y and the simple real reflections for y. That is, we may study it assuming that G is split and that y is attached to the split Cartan.

We now look at how this proposition applies to two special cases.

Corollary 2.3. *Suppose $y_0 \in X$, and that every element of S is imaginary for y_0. (This means that y_0 is attached to the compact Cartan in an equal rank group.) Suppose $x \in X$ is not attached to the compact Cartan.*

1. *If there is a complex s-descent x' for x, then $y_0 < x$ if and only if either $y_0 < x'$ or $s \times y_0 < x'$.*

2. *If there is no complex descent for x, then $y_0 < x$ if and only if there is a real s-descent x' of x such that $y_0 < x'$.*

This is immediate. Again in the second case, the interval below x is contained in the subset of X generated by x and the simple real real reflections for x. We may study the interval by assuming that G has both a compact and a split Cartan, and that x is attached to the split Cartan.

Corollary 2.4. *Suppose $y_1 \in X$, and that every element of S is real for y_1. (This means that y_1 is attached to the split Cartan of a split group.) Suppose $x \in X$ is not attached to the split Cartan.*

1. *If there is a complex s-ascent x' for x, then $y_1 > x$ if and only if either $y_1 > x'$ or $s \times y_1 > x'$.*

2. *If there is no complex ascent for x, then $y_1 > x$ if and only if there is an imaginary s-ascent x' of x such that $y_1 > x'$.*

This is the preceding corollary applied to the dual block. In the second case the interval above x is contained in the subset of X generated by x and the imaginary reflections for x. We may study the interval by assuming that G has both a split and a compact Cartan, and that x is attached to the split Cartan.

3 New algorithm

The induction is first of all increasing on y in the Bruhat order, and then (for a single fixed y) by decreasing induction on x in the Bruhat order. The first step in the algorithm is exactly as at present: we seek a descent for y that is either complex or real type I; this is a direct recursion as above, and the formula for $P_{x,y}$ involves only $P_{*,z}$ with z either an s-descent of y, or else below such an s descent in the Bruhat order (and therefore strictly below y).

We may therefore assume that no such direct recursion exists; or (a weaker assumption) that

$$\text{each descent of } y \text{ is real or compact imaginary.} \qquad \text{(split1)}$$

Remember that in this case the Bruhat interval below y lives inside the subset of X generated by the simple real roots for y. Essentially we can think that

$$G \text{ is split, and } y \text{ is attached to the split Cartan.} \qquad \text{(split2)}$$

(Computationally this just means that we make use only of the simple roots that are real for y.) We want to compute $P_{x,y}$, assuming that we know all $P_{z,y}$ with $z > x$ and all P_{z_1,y_1} with $y_1 < y$. We may assume $x \leq y$ (or else the polynomial is zero). If $x = y$ the polynomial is 1; so assume $x < y$. Then Corollary 2.4 says

Lemma 3.1. *Under the hypotheses (split1) and (split2), suppose that $x < y$. Then there is an $s \in S$ (real for y) such that either*

1. *s is* `C+` *for x, with s-ascent x'; or*
2. *s is* `i2` *for x, with s-ascents $\{x', x''\}$, and $x' \leq y$; or*
3. *s is* `i1` *for x, with s-ascent $x' \leq y$.*

(In case 1), we don't care whether x' is less than y or not; if it is not, then the polynomial $P_{x',y}$ that we need below is zero.)

If s is a descent for y, then the "easy recursion" of (1.9) applies to compute $P_{x,y}$. We may therefore assume henceforth that

$$\text{each complex or imaginary ascent } s \text{ for } x \text{ is } \texttt{rn} \text{ for } y. \qquad \text{(split3)}$$

Using the lemma, we fix such an ascent s for x. Formula (6.15) of [3] says

$$(T_s + 1)C_y = \sum_{z<y,\ s\in\tau(z)} \mu(z,y)q^{(\ell(y)-\ell(z)+1)/2}C_z. \tag{3.2}$$

The coefficient of x on the left side is

$$\begin{cases} P_{x,y} + qP_{x',y} & (s\ \texttt{C+}\text{ for } x)\\ 2P_{x,y} + (q-1)(P_{x',y} + P_{x'',y}) & (s\ \texttt{i2}\text{ for } x)\\ P_{x,y} + P_{s\times x,y} + (q-1)P_{x',y} & (s\ \texttt{i1}\text{ for } x). \end{cases} \tag{3.3}$$

The coefficient of x on the right side of (3.2) is

$$\sum_{\substack{x<z<y\\ s\in\tau(z)}} \mu(z,y)q^{(\ell(y)-\ell(z)+1)/2}P_{x,z}. \tag{3.4}$$

In cases (1) and (2), this leads to a straightforward recursion formula for $P_{x,y}$; for example, if s is $\texttt{i2}$ for x, we get

$$\begin{aligned} 2P_{x,y} = &-(q-1)(P_{x',y} + P_{x'',y})\\ &+ \sum_{\substack{x<z<y\\ s\in\tau(z)}} \mu(z,y)q^{(\ell(y)-\ell(z)+1)/2}P_{x,z}, \qquad (s\ \texttt{i2}\text{ for } x) \end{aligned} \tag{3.5}$$

All the terms on the right are known by inductive hypothesis (including $\mu(z,y)$, which is a coefficient of some $P_{z,y}$ with $z > x$).

$$\begin{aligned} P_{x,y} + P_{s\times x,y} = &-(q-1)P_{x',y}\\ &+ \sum_{\substack{x<z<y\\ s\in\tau(z)}} \mu(z,y)q^{(\ell(y)-\ell(z)+1)/2}P_{x,z}, \qquad (s\ \texttt{i1}\text{ for } x) \end{aligned} \tag{3.6}$$

We can make a parallel analysis when s is $\texttt{ic}$ for x and $\texttt{rn}$ for y. In this case the coefficient of x on the left side of (3.2) is $(q+1)P_{x,y}$, and on the right it is

$$\sum_{z<y,\ s\in\tau(z)} \mu(z,y)q^{(\ell(y)-\ell(z)+1)/2}P_{x,z}.$$

We therefore get a formula

$$(q+1)P_{x,y} = \mu(x,y)q^{(\ell(y)-\ell(x)+1)/2} + \sum_{\substack{x<z<y \\ s\in\tau(z)}} \mu(z,y)q^{(\ell(y)-\ell(z)+1)/2}P_{x,z}. \tag{3.7}$$

We know the sum by inductive hypothesis; but the first term on the right is by definition the leading term on the left. The conclusion is that the formula (3.7) determines every coefficient of $(q+1)P_{x,y}$ except the one of highest degree. But if we use the fact that $(q+1)P_{x,y}$ must vanish at $q=-1$, we can then compute the highest degree term (as the alternating sum of the remaining terms).

These recursions fail to compute $P_{x,y}$ only under the following conditions on $x<y$:

1. there are no complex descents for y;
2. among the s which are real for y, there are no complex ascents for x;
3. among the s which are real for y and imaginary for x, each `r1` or `r2` descent for y is an `ic` descent for x.
4. among the s which are real for y and imaginary for x, each `rn` ascent for y is an `i2` or `i1` ascent for x.

Essentially the first two conditions allow us to reduce to the case that G is split and equal rank, y is attached to the split Cartan, and x is attached to the compact Cartan. Then the last two conditions mean that nonparity simple roots for y correspond precisely to noncompact simple roots for x. (The "direct recursion" for an `r1` descent of y and for an `i2` ascent of x allow us to assume also that the descents for y are all `r2`, and the ascents of x are all `i1`. But this additional restriction is not very useful or important.)

Here's a way to deal with these remaining cases. It's along the lines of `Thicket` but enormously simplified because of the new recursion formulas. Because the block has both a compact and a split Cartan (and $x \neq y$) there must be some real descents for y, which means that there are some `ic` simple roots for x. Except in G2, the long simple roots cannot all be compact in the split form; so

there is a "long" `i1` or `i2` s for x adjacent to an `ic` t for x; (endgame)

here "long" means "long except in G2." Write α for the simple root corresponding to s, and β for the simple root corresponding to t, in some positive system related to the parameter x. The assumptions in (endgame) mean that

$$\alpha \text{ is noncompact and } \beta \text{ is compact.}$$

Because β is "long" and adjacent to α,

$$s(\beta) = \beta + (\text{odd multiple of } \alpha),$$

Therefore $s(\beta)$ is a noncompact root, which means that the status of t for $s \times x$ is noncompact imaginary.

Here is a table of the status of s and t:

block elt	s	t
x	`i1` or `i2`	`ic`
$s \times x$	`i1` or `i2`	`i1` or `i2`
y	`rn`	`r2`

(3.8)

(The second column shows that $s \times x \neq x$, and therefore that s must in fact be `i1` for x.) Then t gives a direct recursion to compute $P_{s\times x,y}$, and s gives through (3.6) a formula for $P_{x,y} + P_{s\times x,y}$. Then $P_{x,y}$ is the difference.

Final comment (DV): I think I proved that if G is equal rank and split, X is the big block, $x \in X$ is attached to the compact Cartan, and $y \in X$ is attached to the split Cartan, then necessarily $x < y$. But of course $P_{x,y}$ might be zero.

References

[1] Fokko du Cloux and Jeffrey Adams, *Implementation of the Kazhdan-Lusztig algorithm* (2011).

[2] David A. Vogan Jr., *The Kazhdan-Lusztig conjecture for real reductive groups*, Representation theory of reductive groups (Park City, Utah, 1982), Progr. Math., vol. 40, Birkhäuser Boston, Boston, MA, 1983, pp. 223–264.

[3] ———, *Irreducible characters of semisimple Lie groups III. Proof of the Kazhdan-Lusztig conjectures in the integral case*, Invent. Math. **71** (1983), 381–417.