Strong real forms and the Kac classification

Jeffrey Adams

October 21, 2005

This paper is expository. It is a mild generalization of the Kac classification of real forms of a simple Lie group to strong real forms. The basic reference for strong real forms in this language is [1]. For the Kac classification we follow [6]. There is also a treatment in [3], in slightly different terms.

1 Real forms and strong real forms

Let $G$ be a reductive algebraic group. We will occasionally identify algebraic groups with their complex points. We have the standard exact sequence

\[
1 \to \text{Int}(G) \to \text{Aut}(G) \to \text{Out}(G) \to 1
\]

where $\text{Int}(G) \cong G/Z(G)$ is the group of inner automorphisms of $G$, $\text{Aut}(G)$ is the automorphims of $G$, and $\text{Out}(G) = \text{Aut}(G)/\text{Int}(G)$.

**Definition 1.2**

1. A real form of $G$ is an equivalence class of involutions in $\text{Aut}(G)$, where equivalence is by conjugation by $G$, i.e. the action of $\text{Int}(G)$.

2. A traditional real form of $G$ is an equivalence class of involutions, where equivalence is by the action of $\text{Aut}(G)$.

The real form defined by $\theta$ has a maximal compact subgroup whose complexification is $K = G^\theta$. 

1
Remark 1.3 A real forms are also defined by an antiholomorphic involution \( \sigma \) of \( G(\mathbb{C}) \), i.e. \( G(\mathbb{R}) = G(\mathbb{C})^\sigma \). Given \( \theta \) choose an antiholomorphic involution \( \sigma_0 \) so that \( \theta \sigma_0 = \sigma_0 \theta \) and \( G(\mathbb{R})_0^\sigma \) is compact. Then the real form defined by \( \theta \) is given by \( \sigma = \theta \sigma_0 \). See [4, Section VI.2].

We say two involutions \( \theta, \theta' \in \text{Aut}(G) \) are inner to each other, or in the same inner class, if they have the same image in \( \text{Out}(G) \). Such a class is determined by an involution \( \gamma \in \text{Out}(G) \), and we refer this inner class as the real forms of \((G, \gamma)\).

We will work entirely in a fixed inner class, so fix an involution \( \gamma \in \text{Out}(G) \).

Fix a splitting datum for the exact sequence (1.1). This is a set \((H, B, \{X_\alpha\})\) consisting of a Cartan subgroup \( H \), a Borel subgroup \( B \) containing \( H \), and a set of simple root vectors. This induces a splitting \( \text{Out}(G) \to \text{Aut}(G) \) of (1.1), and we let \( \theta \) be the image of \( \gamma \) in \( \text{Aut}(G) \). Thus \( \theta \) is an involution of \( G \), it corresponds to the “most compact” real form in the given inner class. We let \( K = G^\theta \).

Remark 1.4 Suppose \( G \) is simple and simply connected. It does not necessarily follow that \( K \) is simply connected; it is not simply connected if and only if the real form \( G = G(\mathbb{R}) \) of \( G \) corresponding to \( K \) has a non-linear cover. Since \( \theta \) is the most compact inner form of \((G, \gamma)\) \( K \) has a chance to be simply connected. In fact this holds unless \( G = SL(2n+1) \), in which case \( K = SO(2n+1) \) and \( \pi_1(K) = \mathbb{Z}/2\mathbb{Z} \). This exception is due to the fact that \( \Delta_\theta \) (cf. Lemma B.1) is not reduced in this case. See the table in Section 3.1.

For most of these notes \( G \) will be semisimple, or even simple. Let \( \Delta = \Delta(G, H) \) be the root system of \( H \) in \( G \), and let \( D = D(\Delta) \) be the Dynkin diagram of \( \Delta \). A choice of splitting datum induces an isomorphism

\[
\text{Out}(\mathfrak{g}) \simeq \text{Aut}(D)
\]

Furthermore \( \text{Out}(G) \subset \text{Out}(\mathfrak{g}) \), with equality if \( G \) is simply connected or adjoint. Thus \( \gamma \) is given by an involution of \( D \).

Let

\[
G^\Gamma = G \rtimes \langle \delta \rangle
\]

where \( \delta^2 = 1 \) and \( \delta g \delta^{-1} = \theta(g) \).
**Definition 1.6** A strong real form of \((G, \gamma)\) is an equivalence class of elements \(x \in G^\Gamma\), satisfying \(x \not\in G\), and \(x^2 \in Z(G)\), where equivalence is by conjugation by \(G\).

The map \(x \to \theta_x = \text{int}(x)\) is a surjection from the strong real forms of \((G, \gamma)\) to the real forms of \((G, \gamma)\). Let

\[ H^\Gamma = H \rtimes \langle \delta \rangle \subset G^\Gamma. \]

Let \(T\) be the identity component of \(H\θ\), and \(A\) be the identity component of \(H^{-\theta} = \{ h \in H \mid \theta(h) = h^{-1} \}\). Then \(H = TA\). Let

\[ T^\Gamma = T \times \langle \delta \rangle \subset H^\Gamma. \]

**Remark 1.7** We may write

\[ (1.8) \quad H \simeq \mathbb{C}^a \times \mathbb{C}^b \times (\mathbb{C}^* \times \mathbb{C}^*)^c \]

where \(\theta\) acts trivially on the first \(a\) factors, by inverse on the next \(b\), and \(\theta(z, w) = (w, z)\) on each of the last \(c\) terms. Note that if \(b \neq 0\) then \(T\) is a proper subset of \(H\θ\). This happens, for example, in \(SO(3, 1)\).

A key observation is that every element of \(H\delta\) is conjugate to an element of \(T\delta\), since \(A\) acts by conjugation on \(H\delta\) by multiplication by \(A\). That is for \(a \in A, h \in H\), \(a(h\delta)a^{-1} = ah(\delta a^{-1})\delta = ah\theta(a^{-1})\delta = a^2 h\delta\). Since \(A\) is connected every element has a square root, so this gives multiplication by an arbitrary element of \(A\). Therefore \(ta\delta\) is conjugate to \(t\delta \in T\delta\). (In fact we could replace \(T\) with \(H/A = T/T \cap A\), see ?.)

**Lemma 1.9** Suppose \(x \in G\delta\) is a semi-simple element (i.e. \(x = g\delta\) with \(g \in G\) semisimple). Then \(x\) is \(G\)-conjugate to an element of \(T\delta\).

**Proof.** Write \(x = g\delta\) and choose a Cartan subgroup \(H'\) containing \(g\). Write \(H' = T'A'\) as usual and \(g = ta\) accordingly. As above we may assume \(a = 1\). Since \(T\) is a Cartan subgroup of \(K\) we may choose \(k \in K\) so that \(ktk^{-1} \in T\), and then \(k(t\delta)k^{-1} \in T\delta\). 

Let \(W = \text{Norm}_G(H)/H\). Then \(\theta\) acts on \(W\), and we let \(W^\theta\) be its fixed points. Note that \(W^\theta\) acts naturally on \(T\), \(A\) and \(T \cap A\).

**Lemma 1.10**

\[ (1.11) \quad W(K^0, T) \simeq W(G, H)^\theta \]
**Remark 1.12** In almost all cases $K$ is connected, and $T$ is a Cartan subgroup of $K$. If $K$ is not connected then $H^\theta$ is a Cartan subgroup of $K$. In this case $H^\theta = TZ(G)$, and $W(K, H^\theta) \simeq W(K^0, T)$. This is the case, for example, if $G = SO(2n)$ and $K = S[O(2n - 1) \times O(1)] \simeq O(2n - 1)$.

**Remark 1.13** One consequence of Lemma 1.10 is this: if $w \in W^\theta$ we may choose a representative $g \in \text{Norm}_G(H)$ of $w$ to be in $K$.

**Lemma 1.14** Suppose $x, x' \in T^\delta$ are $G$-conjugate. Then there exists $g \in \text{Norm}_G(T^\delta)$ so that $gxg^{-1} = x'$.

Thus $G$-conjugacy of elements of $T^\delta$ is controlled by the group $W_\delta$ of the next definition.

**Definition 1.15**

\[(1.16)\]

\[W_\delta = \text{Norm}_G(T^\delta)/\text{Cent}_G(T^\delta)\]

It is well known that $\text{Cent}_G(T) = H$. Therefore $\text{Norm}_{K^0}(T) \subset \text{Norm}_G(H)$ and we obtain a map

\[W(K^0, T) \hookrightarrow W(G, T)\]

whose image is contained in $W(G, T)^\theta$.

**Proposition 1.17**

\[(1.18)\]

\[W_\delta \simeq W^\theta \ltimes (A \cap T).\]

The subgroup $W^\theta$ is the stabilizer of $\delta$ in $W_\delta$, and acts on $T$ via its natural action. The subgroup $A \cap T$ acts on $T^\delta$ by multiplication.

**Proof.** It is well known that $\text{Cent}_G(T) = H$ (every root $\alpha \in \Delta(G, H)$ is non-trivial on $T$, since there are no real roots). Therefore $\text{Norm}_G(T) = \text{Norm}_G(H)$. Thus

\[(1.19)(a)\]

\[\text{Norm}_G(T^\delta) = \{g \in \text{Norm}_G(H) \mid g\delta g^{-1} \in T\}.\]

It is also clear that

\[(1.19)(b)\]

\[\text{Cent}_G(T) = \{g \in H \mid g\delta g^{-1} = \delta\} = H^\theta\]
Therefore
\[(1.19)(c) \quad W_\delta = \{ g \in \text{Norm}_G(H) \mid g\delta g^{-1} \in T \} / H^\theta. \]

We also have
\[(1.19)(d) \quad W^\theta = \{ g \in \text{Norm}_G(H) \mid g\delta g^{-1} \in H \} / H \]

If \( g\delta g^{-1} = ta \in H \), choose \( b \in H \) so that \( b^2 = a \). Then \( (bg)\delta(bg)^{-1} = t \in T \).

It follows that the natural map \( W_\delta \to W^\theta \) is a surjection. The kernel is
\[(1.20) \quad \{ h = ta \in H \mid a^2 \in T \} / H^\theta \]

Let \( A_0 = \{ a \in A \mid a^2 \in T \} \), so the kernel is
\[(1.21) \quad TA_0 / H^\theta = TA_0 / TA^\theta = A_0 / A^\theta. \]

Now the map \( a \to a^2 \) takes \( A_0 \) onto \( A \cap T \) and there is an exact sequence
\[(1.22) \quad 1 \to A^\theta \to A_0 \to A \cap T \to 1 \]

Therefore \( A_0 / A^\theta \simeq A \cap T \). See Remark 1.24.

Putting this together we have an exact sequence
\[(1.23) \quad 1 \to A \cap T \to W_\delta \to W^\theta \to 1 \]

Define a splitting of (1.23) by taking \( w \in W^\theta \) to the unique preimage in \( W_\delta \) fixing \( \delta \). This exists by Lemma 1.10: given \( w \in W^\theta \) there exists \( g \in \text{Norm}_K(H) \subset \text{Norm}_G(H) \) representing \( w \). It is easy to see this is a well defined splitting.

The action of \( W^\theta \) on \( T\delta \) is clear. For \( a \in A \cap T \) choose \( b \in A_0 \) so that \( b^2 = a \). Then \( b(t\delta)b^{-1} = bt\theta(b)^{-1}\delta = b^2t\delta = a(t\delta) \), so \( A \cap T \) acts by multiplication. \( \blacksquare \)

**Remark 1.24** With respect to the decomposition (1.8) we have
\[
A_0 \simeq (\mathbb{Z}/2\mathbb{Z})^b \times (\mathbb{Z}/4\mathbb{Z})^c
\]
\[
A^\theta \simeq (\mathbb{Z}/2\mathbb{Z})^b \times (\mathbb{Z}/2\mathbb{Z})^c
\]
\[
A \cap T \simeq 1 \times (\mathbb{Z}/2\mathbb{Z})^c
\]

where \( \mathbb{Z}/4\mathbb{Z} = \{ \pm(1, 1), \pm(i, -i) \} \subset \mathbb{C}^* \times \mathbb{C}^* \). This makes (1.22) explicit.
**Proposition 1.25** The strong real forms of \((G, \gamma)\) are are parametrized by elements \(x\) of \(T\delta\) satisfying \(x^2 \in Z\), modulo the action of \(W_\delta\).

It is convenient to mod out by the translations in \(T \cap A\); this amounts to replacing \(T\) with \(H/A \cong T/T \cap A\). Let

\[
T = T/T \cap A, \quad T^\Gamma = T \times \langle \delta \rangle
\]

(1.26)

Note that \(W^\theta\) acts on \(T\). Also every element of \(T \cap A\) has order 2, so the condition \(x^2 \in Z\) for \(x \in \overline{T}\) is well defined. This gives:

**Proposition 1.27** The strong real forms of \((G, \gamma)\) are are parametrized by elements \(x\) of \(T\delta\) satisfying \(x^2 \in Z\), modulo the action of \(W^\theta\).

One advantage of \(T\delta\) over \(T\delta\) is that \(Z\) acts naturally on \(T\), via the isomorphism \(T \cong H/A\).

To compute the orbits of \(W_\delta\) on \(T\delta\) we pass to the tangent space, in which \(W_\delta\) becomes an affine Weyl group. See the Appendix for some generalities about affine root systems and Weyl groups.

## 2 Affine Weyl group and strong real forms

We are interested in computing the orbits of \(W^\theta\) acting on \(T\delta\) (Proposition 1.25).

Let \(\pi : E \to T\delta\) be the tangent space of \(T\delta\) at \(\delta\). We recall a few definitions from the Appendix. The space \(E\) is an affine space, with group of translations \(t = \text{Lie}(T)\). The space of affine linear functions \(E \to E\) is denoted \(\text{Aff}(E, E)\).

**Definition 2.1** Suppose \(B\) is a subgroup of \(\text{Aut}(T\delta)\). Let \(\tilde{B}\) be the lift of \(B\) to \(\text{Aff}(E, E)\). That is

\[
\tilde{B} = \{ \phi \in \text{Aff}(E, E) \mid \phi \text{ factors to an element of } B \}.
\]

From Proposition 1.27 we see:

**Lemma 2.2** Strong real forms of \(G\) are parametrized by elements \(X\) of \(E\) satisfying \(\pi(X)^2 \in Z\) modulo the action of \(\tilde{W}^\theta\).
We consider the problem of finding a fundamental domain for the action of $\tilde{W}^\theta$ on $E$, and return later to the question of finding the subset of $X$ such that $\pi(X)^2 \in Z$.

We first suppose $G$ is simply connected. From the Appendix (Definitions B.7 and B.9 and Proposition B.12)

$$\tilde{W}^\theta = W_{\text{aff}} \simeq W^\theta \rtimes L_{\text{sc}}$$

(the last isomorphism depending on a choice of $\tilde{\delta}$ lying over $\delta$). Also $W_{\text{aff}}$ is the affine Weyl group of the affine root system $D_{\text{Aff}}$. The underlying finite root system is $\Delta_{\theta}$.

There is a standard choice of a fundamental domain for the action of $W_{\text{aff}}$ on $E$. Choose a set of simple roots $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_n$ of $\Delta_{\text{aff}}$, and let

$$\bar{D} = \{ e \in E | \tilde{\alpha}_i(e) \geq 0, i = 0, \ldots, n \}.$$ 

If we choose $\tilde{\delta}$ then we may identify $E$ with $V$, and write $\tilde{\alpha}_i = (\alpha_i, 0) (i = 1, \ldots, n)$ and $\tilde{\alpha}_0 = (\alpha_0, c)$. Let $\beta = -\alpha_0$; recall $\beta$ is the highest long (resp. short) root of $\Delta$ if $c = 1$ (respectively $c = 2$). Then

$$\bar{D} = \{ v \in V | \alpha_i(v) \geq 0 (i = 1, \ldots, n), \beta(v) \leq c \}.$$

If $G$ is not simply connected then $W_{\text{aff}} \subset \tilde{W}^\theta$, and $\tilde{W}^\theta$ is an extended affine Weyl group. Its fundamental domain will be a quotient of $D$ by a finite group.

**Definition 2.3**

Let

$$L(G) = X_*(T/T \cap A).$$

In particular we have

$$L(G)/X_*(T) \simeq T \cap A$$

**Lemma 2.6**

$$L(G) = \langle \frac{1}{c} \sum_{k=0}^{c-1} \theta^k(\gamma^\vee) | \gamma \in X_*(H) \rangle$$

7
If \( c = 1, 2 \) we have

\[
L = \left\{ \frac{1}{2}(\alpha^\vee + \theta \alpha^\vee) \mid \alpha \in X_*(H) \right\} \quad (c = 1, 2).
\]

If \( G \) is simply connected then \( L(G) = L_{sc} \) (Definition B.9).

**Lemma 2.8** Setting \( L = L(G) \) we have an exact sequence

\[
(2.9)(a) \quad 1 \to L \to \tilde{W}^\theta \to W^\theta \to 1
\]

Given \( \tilde{\delta} \) we obtain a splitting of (2.9)(a), so

\[
(2.9)(b) \quad \tilde{W}^\theta \cong W^\theta \ltimes L.
\]

If \( G \) is simply connected then (2.9)(a-b) reduce to (B.13)(a-b).

To find a fundamental domain for \( \tilde{W}^\theta \) we relate it to \( W_{aff} \).

**Lemma 2.10** We have an exact sequence

\[
(2.11) \quad 1 \to W_{aff} \to \tilde{W}^\theta \to L/L_{sc} \to 1
\]

Given \( \tilde{\delta} \) we obtain a splitting taking \( L/L_{sc} \) to the stabilizer of \( D \). Thus

\[
(2.12) \quad \tilde{W}^\theta \cong W_{aff} \ltimes L/L_{sc}
\]

and \( L/L_{sc} \) acts as automorphisms of \( D \).

Recall we are given \((\Delta, \theta)\), to which we have associated the affine root system \( \Delta_{aff} \), with Dynkin diagram \( D_{Aff} \). See the Appendix.

**Lemma 2.13** The stabilizer of \( D \) in the Euclidean group of \( E \) is isomorphic to the automorphism group of \( D_{Aff} \).

Thus we have an action of \( L/L_{sc} \) on \( D_{Aff} \). It behooves us to understand \( L/L_{sc} \).
2.1 The group $L/L_{sc}$

From (B.11) we have

$$L/L_{sc} = \frac{\langle \{ \frac{1}{2}(\gamma^\vee + \theta \gamma^\vee) | \gamma^\vee \in X_s(H) \} \rangle}{\langle \{ \frac{1}{2}(\alpha^\vee + \theta \alpha^\vee) | \gamma^\vee \in R^\vee \} \rangle}$$

Let $G_{sc}$ be the simply connected cover of $G$, with center $Z_{sc} = Z(G_{sc})$. We have an exact sequence

$$1 \to \pi_1 \to G_{sc} \to G \to 1$$

with $\pi_1 = \pi_1(G) \subset Z_{sc}$. Write $H_{sc} = T_{sc}A_{sc}$ for the Cartan subgroup in $G_{sc}$ with image $H$.

**Lemma 2.14**

(2.15) $$L/L_{sc} \simeq \pi_1/\pi_1 \cap A_{sc}$$

**Proof.** A standard fact is that $\pi_1 \simeq X_s(H)/R^\vee$. The map $\gamma^\vee \to \frac{1}{2}(\gamma^\vee + \theta \gamma^\vee)$ takes $X_s(H)$ onto $L$ and factors to a surjection

$$\pi_1 \to L/L_{sc}.$$ 

The kernel is

$$\{ \gamma^\vee \in X_s(H) | (1 + \theta)\gamma^\vee \in (1 + \theta)R^\vee \} / R^\vee$$

If $(1 + \theta)\gamma^\vee = (1 + \theta)\mu^\vee$ for some $\mu^\vee \in R^\vee$ then $(1 + \theta)(\gamma^\vee - \mu^\vee) = 0$. So we may replace the numerator with $\{ \gamma^\vee | (1 + \theta)\gamma^\vee = 0 \}$. This says $\exp(2\pi i \gamma^\vee) \in A_{sc}$, so the kernel is $\pi_1 \cap A_{sc}$. □

**Remark 2.16** Note that

$$(1 - \theta)\pi_1 \subset \pi_1 \cap A_{sc} \subset \pi_1^{-\theta}$$

and both inclusions may be proper. If $G$ is adjoint then $\pi_1 = Z_{sc}$ and one can see $Z_{sc} \cap A_{sc} = (1 - \theta)Z_{sc}$, which gives

(2.17) $$L_{ad}/L_{sc} = Z_{sc}/(1 - \theta)Z_{sc}.$$ 

However it is not easy to describe $\pi_1 \cap A_{sc}$ in general.
Definition 2.18  Let

\begin{equation}
\pi_1^\dagger = \pi_1 / \pi_1 \cap A_{sc}
\end{equation}

Let \( \tau : \pi_1^\dagger \to \text{Aut}(D_{Aff}) \) be the action of \( \pi_1^\dagger \) on the affine Dynkin diagram via Lemmas 2.10, 2.13 and (2.15).

Here is another description of \( \tau \). First take \( G \) to be simply connected, so \( Z = Z_{sc} \). Note that \( Z \) acts by left multiplication on \( H\delta \) and therefore on \( T\delta \). Explicitly \( z = ta \in Z \) acts on \( T\delta \) by multiplication by \( t \). Although \( t, a \) are only defined up to \( T \cap A \), this action is well defined on \( T\delta \). Clearly this action factors to \( Z/Z \cap A \), lifts to an action on \( E \), and induces actions of \( Z/Z \cap A \) on \( D \) and \( D_{Aff} \).

Suppose \( z = ta = \exp(2\pi i \gamma^\vee) \) with \( \gamma^\vee \in P^\vee \). Then \( t = \exp(2\pi i \frac{1}{2}(\gamma^\vee + \theta \gamma^\vee)) \), and it follows that under the isomorphism (2.15) \( L_{ad}/L_{sc} \) acts by translation on \( E \).

Now drop the assumption that \( G \) is simply connected. Then \( \pi_1(G) \subset Z_{sc} \) acts on \( D \) and \( D_{Aff} \) by the preceding construction, and this action factors to an action of \( \pi_1^\dagger(G) \).

Lemma 2.20 We may parametrize \( \overline{D} \) as \( \{(a_0, \ldots, a_n)\} \) where \( a_i \geq 0 \) and

\begin{equation}
\sum_{i=0}^{n} n_i a_i = \frac{1}{c}.
\end{equation}

Here \( (a_0, \ldots, a_n) \) corresponds to the element \( X \) of \( D \) satisfying

\[ \alpha_i(X) = a_i \quad (i = 1, \ldots, n) \]

Lemma 2.22 Suppose \( (a_0, a_1, \ldots, a_n) \) satisfies (2.21), and let \( X \in D \) be the corresponding element. Then \( x = \pi(X) \in \overline{T\delta} \) satisfies \( x^m \in Z \) if and only if \( ma_i \in Z \) for all \( i = 0, \ldots, n \).

Example 2.23 Take \( m = 1 \). We must take \( c = 1 \) and each \( a_i \) is 0 or 1. We conclude from (2.21) that \( Z \) is in bijection with the nodes of \( \tilde{D} \) with label 1.

Given \( m \) choose integers \( b_i \) and let \( a_i = b_i/m \) (\( 0 \leq i \leq n \)). Then \( (a_0, \ldots, a_n) \) corresponds to an element of \( D \) if
To complete our classification of strong real forms we take \( m = 1 \) or \( 2 \).

**Definition 2.25** A Kac diagram for \((G, \gamma)\) is a subset \( S \) of \( D_{\text{Aff}} \) satisfying
\[
c \sum_{i \in S} n_i b_i = m
\]
Clearly \(|S| \leq 2\) and \( n_i \leq 2 \) for all \( i \in S \).

**Theorem 2.26** Fix \( G \) and an inner class \( \gamma \) of real forms. Let \( c = \text{order}(\gamma) \). Let \( \theta \) be the fundamental real form in the given inner class. Let \( \Delta \) be the root system of \( G \), \( \Delta_\theta \) the quotient of \( \Delta \) by \( \theta \), and \( D_{\text{Aff}} \) the affine Dynkin diagram associated to \( \Delta_\theta \).

The strong real forms of \((G, \gamma)\) are parametrized by Kac diagrams for \( D_{\text{Aff}} \), modulo the action of \( \pi_1^*(G) \) on \( D_{\text{Aff}} \).

Suppose \( S \) is a Kac diagram corresponding to a real form, with (complexified) maximal compact subgroup \( K_S \). Then the Dynkin diagram of \( K_S \) is obtained by by deleting the nodes of \( S \) from \( D_{\text{Aff}} \).

For the usual classification of real forms see the next section.

For example, a compact group is given by \( m = 1, c = 1 \) and \( S = \{i\} \) with \( n_i = 1 \).

Suppose \( m = 2 \). If \( c = 1 \), then \( S = \{i\} \) with \( n_i = 2 \), or \( S = \{i, j\} \) with \( n_i = n_j = 1 \). If \( c = 2 \) then \( S = \{i\} \) with \( n_i = 1 \).

### 2.2 The Kac classification of real forms

The Kac classification of real forms of \( \mathfrak{g} \) amounts to taking \( G \) to be the adjoint group. In this case \( \pi_1^*(G) = Z_{\text{sc}}/(1 - \theta)Z_{\text{sc}} \) (2.17). Recall (2.19) acts by \( \tau \) on \( D_{\text{Aff}} \) (Definition 2.18).

**Theorem 2.27** Traditional real forms of \((\mathfrak{g}, \gamma)\) are parametrized by subsets \( S \) as in Theorem 2.26, modulo the action of \( \text{Aut}(D_{\text{Aff}}) \).

Real forms of \((\mathfrak{g}, \gamma)\) are parametrized by subsets \( S \) modulo the action of \( Z_{\text{sc}}/(1 - \theta)Z_{\text{sc}} \).
Proof. The second statement is an immediate consequence of Theorem 2.26. The first follows from the following Lemma. ■

Remark 2.28 This also gives the classification for $G$ either simply connected or adjoint. For general $G$ equivalence will be by the subgroup stabilizing $Z(G)$.

Lemma 2.29 We have a split exact sequence

$$1 \to \pi^1_1(G) \to Aut(D) \to Out(G) \to 1$$

or equivalently

$$1 \to \pi^1_1(G) \to Aut(D_{\text{Aff}}) \to Aut(D_{\theta}) \to 1$$

Here $D_{\theta}$ is the Dynkin diagram of $\Delta_{\theta}$, the underlying finite root system of $D_{\text{Aff}}$. See the Appendix.

Remark 2.30 If $\theta = 1$ and $G$ is simply connected this becomes

$$1 \to Z \to Aut(D_{\text{Aff}}) \to Aut(D) \to 1$$

If $\theta \neq 1$ then $Aut(D_{\theta}) = 1$ and we have

$$\pi^1_1 \simeq Aut(D_{\text{Aff}})$$

See [6, Exercise 15, page 217]. For an explicit formula for the map $Z \to Aut(D_{\text{Aff}})$ see [2, Chapter VI, §2.3, Proposition 6].

3 Simplified Kac Diagrams and Vogan Diagrams

If $\gamma \neq 1$ the classification of real forms via the Kac diagram is quite subtle, due to its use of the extended Dynkin diagram of $\Delta_{\theta}$, rather than that of $\Delta$. Here is a version using the extended Dynkin diagram of $\Delta$.

So fix $(G, \gamma)$ with $G$ simple and $\gamma \neq 1$. Choosing a splitting datum, in particular a Cartan subgroup $H$ we obtain the fundamental automorphism $\theta$ of $G$ as in Section 1. Write $H = TA$ as usual.
For simplicity we assume $G$ is adjoint, so strong real forms and real forms coincide. Suppose $\gamma \neq 1$. By Proposition 1.27 the real forms of $(G, \gamma)$ are parametrized by elements $t \in T$ of order 2 (corresponding to $x = t\delta \in \overline{T}\delta$), modulo $T \cap A$ and conjugation by $W^\theta$.

On the other hand the real forms of $(G, 1)$ are parametrized by elements of $H$ of order 2, modulo conjugation by $G$. If two elements of $t$ are conjugate by $W$ then they are necessarily conjugate by $W^\theta$. If $S$ is the Kac diagram of a real form of $(G, 1)$, then the corresponding element $h$ is in $T$ if and only if $S$ is $\theta$-invariant. This gives a surjective map from

$$\theta \text{- invariant Kac diagrams for } (G, 1) \to \text{strong real forms of } (G, \gamma)$$

This map is not injective: on the left hand side equivalence is by the action of $W^\theta$, and on the right by $W^\theta$ and $A \cap T$. It turns out that if we require that $S$ is pointwise fixed by $\theta$ then we get a bijection.

**Proposition 3.1** Given $(G, \gamma)$ let $D_{\text{Aff}}$ be the extended Dynkin diagram of $\Delta = \Delta(G, H)$. Then real forms of $(G, \gamma)$ are parametrized by Kac diagrams $S$ for which each node of $S$ is fixed by $\theta$, modulo $\text{Aut}(D_{\text{Aff}})$. That is, $S$ is a set of $\theta$-fixed nodes of $D_{\text{Aff}}$, such that $c \sum_{i \in S} n_i \leq 2$.

To be honest there is some case-by-case checking here. One point is this. Suppose $\alpha$ is a complex root, and $n_\alpha = 1$. Then $S = \{\alpha, \theta\alpha\}$ defines an element $t$ of $T$ of order 2, and a real form of $(G, 1)$. It also defines a real form of $(G, \gamma)$, but this one is obtained from another set $S$ which is pointwise fixed.

### 3.1 Vogan Diagrams

We continue to assume $(G, \gamma)$ and $H$ have been fixed, and $\theta$ is the fundamental real form of $G$. Let $D$ be the Dynkin diagram of $G$. Suppose $\theta'$ is a real form of $(G, \gamma)$ and $B$ is a $\theta$-stable Borel subgroup of $G$ containing $H$. Associated to this data is a Vogan Diagram: color each of the $\theta$-fixed nodes of $D$ black if the corresponding imaginary root is non-compact, and white otherwise. See [4, Section VI.8]. Alternatively, let $S$ be the subset (of black nodes) of the $\theta$-fixed nodes of $D$. This gives a map from real forms of $(G, \gamma)$ to Vogan diagrams. This map is not injective: it depends on the choice of $B$. If we choose $B$ to be the “Borel de Siebenthal” choice [4, Theorem 6.96],
i.e. for which at most one simple root is non-compact, then we get a set $S$ with at most one element.

The is closely related to the simplified Kac diagram. Here is the precise statement.

**Proposition 3.2** Suppose $S$ is a modified Kac diagram of a real form. If $S$ contains a node with label 1 we may assume (via the action of $Z_{sc}$) this is the affine node. Deleting this node we obtain a subset of the finite Dynkin diagram. This is the Vogan diagram of the real form.

Conversely suppose $S$ is a Vogan diagram with at most one node, corresponding to a real form of $G$. Also assume it satisfies the condition in the last line of [4, Thoerem6.96]; equivalently the label on this node is $\leq 2$. If $S$ is empty this is the compact form. Suppose $S = \{i\}$. The Kac diagram of this real form is $S \cup \{0\}$ if $n_i = 1$, and $S$ if $n_i = 1$.

**Remark 3.3** One of the subtleties of the Vogan diagram is that we do not need a diagram $S = \{i\}$ if $n_i \geq 3$. The fact that such Kac diagram is not needed is explained by (2.24).

**Appendix: Affine root systems and Weyl groups**

Let $V$ be a real vector space of dimension $n$ and $E$ an affine space with translations $V$. That is $V$ acts simply transitively on $E$, written $v, e \rightarrow v + e$. A function If $E, E'$ are affine spaces a function $f : E \rightarrow E'$ is said to be affine if there exists a linear function $df : V \rightarrow V'$ such that

\[(A.1) \quad f(v + e) = df(v) + f(e) \quad \text{for all } v, e \in E.\]

In particular if $E'$ is one dimensional we say $f$ is an affine linear functional. In this case $df : V \rightarrow \mathbb{R}$, i.e. $df \in V^*$. We say $df$ is the differential of $f$. The set $\text{Aff}(E)$ of all affine linear functionals is a vector space of dimension $n + 1$. The map $f \rightarrow df$ is a linear map from $\text{Aff}(E)$ to $V^*$, and this gives an exact sequence

\[(A.2) \quad 0 \rightarrow \mathbb{R} \rightarrow \text{Aff}(E) \rightarrow V^* \rightarrow 0.\]
The first inclusion takes $x \in \mathbb{R}$ to the constant function $f_x(e) = x$ for all $e \in E$; this satisfies $df = 0$.

Choose an element $e_0 \in E$. This gives an isomorphism $V \simeq E$ via $v \to v + e_0$. For $\lambda \in V^*$ let $s(\lambda)(v + e_0) = \lambda(v)$. This defines a splitting of (A.2):

**Lemma A.3** Given $e_0$ we obtain an isomorphism

(A.4)(a) \quad \text{Aff}(E) \simeq V^* \oplus \mathbb{R}

According to this decomposition we write $f \in \text{Aff}(E)$ as

(A.4)(b) \quad f = (\lambda, c).

We make the isomorphism (A.4)(a) explicit. In one direction $f \in \text{Aff}(E)$ goes to $\lambda = df$ and $c = f(e_0)$. For the other direction $(\lambda, c)$ goes to $f \in \text{Aff}(E)$ defined by $f(v + e_0) = \lambda(v) + c$.

We now assume $V$ is equipped with a positive definite non-degenerate symmetric form $(,)$, and identify $V$ and $V^*$. In particular we may identify $df$ with an element of $V$. Define $(,)$ on Aff(V) by

$$(f, g) = (df, dg)$$

and for $f \in \text{Aff}(E)$ not a constant function let

$$f^\vee = \frac{2f}{(f, f)}.$$  

The affine reflection $s_f : V \to V$ is

$$s_f(v) = v - f^\vee(v)df$$

$$= v - f(v)(df)^\vee$$

$$= v - \frac{2f(v)}{(f, f)}df$$

**Definition A.5 (Macdonald [5])** An affine root system on $E$ is a subset $S$ of Aff(E) satisfying

1. $S$ spans Aff(E), and the elements of $S$ are non-constant functions,
2. \(s_{\alpha}(\beta) \in S\) for all \(\alpha, \beta \in S\),

3. \(\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}\) for all \(\alpha, \beta \in S\),

4. The Weyl group \(W = W(S)\) is the group generated by the reflections \(\{s_{\alpha} | \alpha \in S\}\). We require that \(W\) acts properly on \(V\).

The Weyl group \(W(S)\) is an affine Weyl group. The notions of simple roots \(\Pi(S)\) and Dynkin diagram \(D(S)\) are similar to those for classical root systems. Also the dual \(S^\vee\) of \(S\) defined in the obvious way is an affine root system, with Dynkin diagram \(D(S^\vee) = D(S)^\vee\). Here the dual of a Dynkin diagram means the same diagram with arrows reversed, as usual.

Choose a base point \(e_0\) in \(E\) and write elements of \(\text{Aff}(E)\) as \((\lambda, c)\) as in Lemma A.3.

Suppose \(\Delta \subset V\) is a classical (not necessarily reduced) root system. If \(\Delta\) is simply laced we say each root is long. Let \(\Pi = \{\alpha_1, \ldots, \alpha_n\}\) be a set of simple roots. For each \(i\) let \(\tilde{\alpha}_i = (\alpha_i, 0)\), and let \(\tilde{\alpha}_0 = (\beta, 1)\) where \(\beta\) is the highest root. Note that \(\beta\) is long. Then \(\{\tilde{\alpha}_0, \ldots, \tilde{\alpha}_n\}\) is a set of simple roots of an affine root system denoted \(\tilde{\Delta}\).

Let \(D = D(\Delta)\) be the Dynkin diagram of \(\Delta\). Let \(\tilde{D}\) be the extended Dynkin diagram of \(D\), i.e. obtained by adjoining \(-\beta\) where \(\beta\) is the highest root. Then the Dynkin diagram of \(\tilde{\Delta}\) is the extended Dynkin diagram of \(\Delta\), i.e.

\[D(\tilde{\Delta}) = \tilde{D}(\Delta).\]

We will use \(\Delta\) (resp. \(S\)) to denote a typical classical (resp. affine) root system.

Suppose \(\Delta\) is a classical root system with Dynkin diagram \(D = D(\Delta)\). Let \(S = \tilde{\Delta}\), so \(D(S) = \tilde{D}\). Then \(S^\vee = (\tilde{\Delta})^\vee\) is also an affine root system, with Dynkin diagram \(D(S^\vee) = (\tilde{D})^\vee\). If \(\Delta\) is not simply laced then it is not necessarily the case that \((\tilde{\Delta})^\vee = (\tilde{\Delta})^\vee\) or \((\tilde{D})^\vee = (\tilde{D})^\vee\). Note that \(\tilde{D}\) is obtained from \(D\) by adding a long root, so \((\tilde{D})^\vee\) has an extra short root. On the other hand \((\tilde{D})^\vee\) is obtained from \(D^\vee\) by adding an extra long root.

**Theorem A.6 (Macdonald [5])** Every reduced, irreducible affine root system is equivalent to either \(\tilde{\Delta}\) or \((\tilde{\Delta})^\vee\) where \(\Delta\) is a classical (not necessarily reduced) root system.
Remark A.7 A remarkable fact is that every reduced, irreducible affine root system is also obtained from a classical root system and involution, as discussed in the next section.

**Affine root system and Weyl group associated to \((\Delta, \theta)\)**

Let \(\Delta\) be an irreducible root system, and \(\theta\) an automorphism of \(\Delta\) preserving a set of simple roots. Thus \(\theta\) corresponds to an automorphism of the Dynkin diagram \(D = D(\Delta)\) of \(\Delta\). Let \(c \in \{1, 2, 3\}\) be the order of \(\delta\). Associated to \((\Delta, \theta)\) is an affine root system, which we now describe.

The quotient \(\Delta/\theta\) is naturally a root system [7], which we denote \(\Delta_\theta\). Here are the possibilities with \(\theta \neq 1\). We list the finite root systems \(\Delta, \Delta_\theta\), the names of the affine root system according to [5] and [6], the simply connected group \(G\) with root system \(\Delta\), the real form of \(G\) corresponding to \(\theta\), and \(G^\theta\).

<table>
<thead>
<tr>
<th>(\Delta)</th>
<th>(\Delta_\theta)</th>
<th>(\Delta_{\text{aff}})</th>
<th>(\Delta_{\text{aff}})</th>
<th>(G)</th>
<th>(G(\mathbb{R}))</th>
<th>(K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_{2n})</td>
<td>(BC_n)</td>
<td>(\widetilde{BC}_n)</td>
<td>(A^{(2)}_{2n})</td>
<td>(SL(2n + 1))</td>
<td>(SL(2n + 1, \mathbb{R}))</td>
<td>(SO(2n + 1))</td>
</tr>
<tr>
<td>(A_{2n-1})</td>
<td>(C_n)</td>
<td>(\widetilde{D}_{n}^{\vee})</td>
<td>(A^{(2)}_{2n-1})</td>
<td>(SL(2n))</td>
<td>(SL(n, \mathbb{H}))</td>
<td>(Sp(n))</td>
</tr>
<tr>
<td>(D_n)</td>
<td>(B_n)</td>
<td>(\widetilde{C}_n^{\vee})</td>
<td>(D^{(2)}_n)</td>
<td>(Spin(2n))</td>
<td>(Spin(2n - 1, 1))</td>
<td>(Spin(2n - 1))</td>
</tr>
<tr>
<td>(E_{6R})</td>
<td>(F_4)</td>
<td>(\widetilde{F}_4^{\vee})</td>
<td>(E^{(2)}_6)</td>
<td>(E_6)</td>
<td>(E_6(F_4))</td>
<td>(F_4)</td>
</tr>
<tr>
<td>(D_4, \ \theta^3 = 1)</td>
<td>(G_2)</td>
<td>(\widetilde{G}_2^{\vee})</td>
<td>(D^{(3)}_4)</td>
<td>(Spin(8))</td>
<td>(G_2)</td>
<td></td>
</tr>
</tbody>
</table>

As in section 1 there is an algebraic group \(G\), and splitting data \((H, B, \{X_\alpha\})\) so that \(\Delta = \Delta(G, H)\), and \(\theta\) may be viewed as an automorphism of \(G\) preserving the splitting data. (For these purposes we may as well take \(G\) simply connected.) Then \(T = H^\theta\) acts on \(\mathfrak{g}\), and the set of roots \(\Delta(G, T) \subset \mathfrak{t}^*\) is a (possibly reduced) root system.

The following Lemma is more or less immediate.

**Lemma B.1** *Restriction from \(H\) to \(T\) defines isomorphisms*

\[
\Delta(G, T) \simeq \Delta_{\theta}
\]

*and*

\[
W^\theta \simeq W(\Delta_{\theta}).
\]
Also \( \Delta(K, T) \) is the reduced root system of \( \Delta_{\theta} \) (obtained by taking only the shorter of two roots \( \alpha, 2\alpha \)) and \( W(K, T) \cong W(\Delta_{\theta}) \). See Remark 1.4.

Now \( T^\Gamma \) acts on the complex Lie algebra \( g \) of \( G \). Let \( \Delta(G, T^\Gamma) \) be the set of roots, i.e. we have a root space decomposition

\[
g = \sum_{\alpha \in \Delta(G, T^\Gamma)} g_\alpha.
\]

Clearly restriction from \( T^\Gamma \) to \( T \) is a surjection \( \Delta(G, T^\Gamma) \to \Delta(G, T) \).

If \( c = 1 \) this is simply \( \Delta(G, T) \). For simplicity assume \( c = 2 \). Then \( \Delta(G, T^\Gamma) \) may be thought of as a \( \mathbb{Z}/2\mathbb{Z} \)-graded root system. That is a character \( \alpha \) of \( T^\Gamma \) is a pair \( (\alpha_0, \epsilon) \) with \( \alpha_0 \in \Delta(G, T) \cong \Delta_{\theta} \) and \( \epsilon = \pm 1 \), where \( \alpha_0 = \alpha|_T \) and \( \epsilon = \alpha(\delta) \). We can define the reflection associated to \( \alpha \in \Delta(G, T^\Gamma) \) in the usual way, preserving \( \Delta(G, T^\Gamma) \). To be precise, if \( \alpha = (\alpha_0, \epsilon) \) and \( \beta = (\beta_0, \delta) \) then

\[
s_\alpha(\beta) = (s_{\alpha_0}(\beta_0), \epsilon\delta(-1)^{\langle \beta, \alpha \rangle} - 1).
\]

Let \( \pi : E \to T\delta \) be the universal cover. Then \( E \) is an affine space with translations \( t = \text{Lie}(t) \).

Suppose \( \lambda \) is a character of \( T^\Gamma \to \mathbb{C}^* \). Note that \( \lambda \) is determined by its restriction to \( T\delta \). By the property of covering spaces \( \lambda \) lifts to a family of functions \( \tilde{\lambda} : E \to \mathbb{C} \) satisfying

\[
\lambda(\pi(X)) = e^{2\pi i \tilde{\lambda}(X)}
\]

i.e. \( d\tilde{\lambda} = d\lambda \), where the left hand side is in the sense of (A.1) and the right is the ordinary differential of \( \lambda \). We say \( \tilde{\lambda} \) lies over \( \lambda \). Any two such functions differ by constant.

**Definition B.3** The affine root system \( \Delta_{\text{aff}} \) associated to \( (\Delta, \theta) \) is the set of affine functions in \( \text{Aff}(E) \) lying over \( \Delta(G, T^\Gamma) \).

Note that the underlying finite root system, i.e. the differentials of all affine roots is \( \Delta(G, T) \cong \Delta_{\theta} \), i.e.

\[
d : \Delta_{\text{aff}} \to \Delta_{\theta}
\]

The following Lemma is an immediate consequence of the fact that \( \Delta(G, T^\Gamma) \) is a root system in the sense of (B.2).
Lemma B.4 \( \Delta_{\text{aff}} \) is an affine root system.

To be explicit, choose \( \tilde{\delta} \in E \) with \( \pi(\tilde{\delta}) = \delta \). Suppose \( \alpha \in \hat{T}^T \). To avoid excessive notation we write \( \alpha \) for the differential of \( \alpha \) restricted to \( T \), rather than \( d\alpha \). Then in the decomposition of Lemma A.3 we may write the set of \( \tilde{\alpha} \) lying over \( \alpha \) as

\[
\{ (\alpha, c) \mid e^{2\pi ic} = \alpha(\delta) \}
\]

In particular note that the set of roots lying over \( \alpha \) is

\[
\{ (\alpha, c) \mid c \in \mathbb{Z} \} \quad \text{if } \alpha(\delta) = 1
\]

or

\[
\{ (\alpha, c) \mid c \in \mathbb{Z} + \frac{1}{2} \} \quad \text{if } \alpha(\delta) = -1
\]

Similarly if \( \delta \) has order 3 then \( c \in \mathbb{Z} + \frac{1}{3} \) or \( \mathbb{Z} + \frac{2}{3} \).

For \( \alpha \in \Delta_\theta \) let \( c_\alpha = 1 \) if \( \alpha \) is long, or \( \frac{1}{c} \) if \( \alpha \) is short, where \( c = \text{order}(\theta) \).

Proposition B.5 Let \( \Delta_{\text{aff}} \) be the affine root system associated to \( (\Delta, \theta) \), and let \( c = \text{order}(\theta) \in \{1, 2, 3\} \). Then

\[
\Delta_{\text{aff}} = \{ (\alpha, x) \mid x \in c_\alpha \mathbb{Z} \}
\]

Proposition B.6 Fix a set \( \alpha_1, \ldots, \alpha_n \) of simple roots of \( \Delta_\theta \). For each \( i \) let \( \tilde{\alpha}_i = (\alpha_i, 0) \). Let \( \beta \) be the highest (long) root of \( \Delta = \Delta_\theta \) if \( c = 1 \) or the highest short root otherwise. Let

\[
\tilde{\alpha}_0 = (-\beta, \frac{1}{c}).
\]

Then \( \{\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\} \) is a set of simple roots of \( \Delta_{\text{aff}} \).

Definition B.7 The affine Weyl group associated to \( (\Delta, \theta) \) is the subgroup of \( \text{Aff}(E, E) \) generated by the affine reflections \( s_{\tilde{\alpha}} \) for \( \tilde{\alpha} \in \Delta_{\text{aff}} \). Alternatively,

\[
W_{\text{aff}} = \{ \phi \in \text{Aff}(E, E) \mid \phi \text{ factors to an element of } W(\Delta_\theta) = W^\theta \}.
\]

We now describe \( W_{\text{aff}} \).

Definition B.9 Let

\[
L_{sc} = \left\{ \frac{1}{c} \sum_{k=0}^{c-1} \theta^k(\alpha^\vee) \mid \alpha \in \Delta \right\}
\]
We are primarily interested in $c = 1, 2$, in which case:

(B.11) \[ L_{sc} = \{ \frac{1}{2}(\alpha^\vee + \theta \alpha^\vee) \mid \alpha \in \Delta \} \]

**Proposition B.12** The lattice $L_{sc}$ is the set of translations in $W_{aff}$. There is an exact sequence

(B.13)(a) \[ 0 \to L_{sc} \to W_{aff} \to W^\theta \to 1 \]

If we choose an element $\tilde{\delta} \in E$ lying over $\delta$ we obtain a splitting of (1.18), taking $W^\theta$ to the the stabilizer in Aff$(E)$ of $\tilde{\delta}$, i.e.

(B.13)(b) \[ W_{aff} \simeq W^\theta \ltimes L_{sc} \]

We give a few details of the map $p : W_{aff} \to W_\delta$. Suppose $\alpha \in \Delta_\theta$ and $x \in \mathbb{Z}$. Then

\[ p(s_{(\alpha,x)}) = s_\alpha. \]

Suppose $c = 2$, $\alpha \in \Delta_\theta$ is a short root and $x \in \mathbb{Z} + \frac{1}{2}$. Then $m_\alpha = \alpha^\vee(-1) \in T \cap A$ and

\[ p(s_{(\alpha,x)}) = s_\alpha m_\alpha \]

and

\[ p(t_{\frac{1}{2}\alpha^\vee}) = m_\alpha. \]

where $t_{\frac{1}{2}\alpha^\vee} \in W_{aff}$ is translation by $\frac{1}{2}\alpha^\vee$.

**References**


