Notes on the Hermitian Dual

Jeffrey Adams

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These notes are incomplete as of 12/22/2008. I'll do more on them after the first of the year.

1 Basics

Let H be a complex torus, with real points $H(\mathbb{R})$, $\mathfrak{h}, \mathfrak{h}_0, \theta, \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ as usual. View $X_*(H)$ as $\frac{1}{2\pi i}\ker(\exp) \subset \mathfrak{h}$.

Write $X \to \overline{X}$ for complex conjugation with respect to $\mathfrak{h}_0$, and $X \to \widetilde{X}$ for the one with respect to $X_*(H) \otimes \mathbb{R}$. Note that $\overline{\theta X} = \theta\overline{X}$ and $\widetilde{\theta X} = \theta\widetilde{X}$, and also

(1) $$\overline{X} = -\widetilde{\theta X} = -\theta\widetilde{X} \quad (X \in \mathfrak{h}).$$

This follows from the fact that $X_*(H) \subset i\mathfrak{t}_0 \oplus \mathfrak{a}_0$ where $\mathfrak{t}_0 = \mathfrak{h}_0^\theta$ and $\mathfrak{a}_0 = \mathfrak{h}_0^{-\theta}$.

Write $\theta^\vee$ for minus-adjoint of θ:

(2) $$\langle \theta X, \lambda \rangle = -\langle X, \theta^\vee \lambda \rangle \quad (X \in \mathfrak{h}, \lambda \in \mathfrak{h}^*).$$

For $\lambda \in \mathfrak{h}^*$ define $\overline{\lambda} \in \mathfrak{h}^*$ by $\overline{\lambda}(X) = \overline{\lambda(\overline{X})}$ (the outer $^-$ is on $\mathbb{C}$), and $\widetilde{\lambda}(X) = \overline{\lambda(\widetilde{X})}$. Then

(3) $$\widetilde{\lambda} = \lambda \quad (\lambda \in X^*(H) \otimes \mathbb{R})$$

and

(4) $$\overline{\lambda} = \theta^\vee(\widetilde{\lambda}) = \widetilde{\theta^\vee(\lambda)} \quad (\lambda \in \mathfrak{h}^*).$$

If χ is a character of $H(\mathbb{R})$, then the *Hermitian dual* of χ is $\chi^h(g) = \overline{\chi(g^{-1})}$.

Let $K = H^\theta$, the complexified maximal compact subgroup of $H(\mathbb{R})$. Suppose (λ, τ) is a $(\mathfrak{h}, K)$-module, i.e. $\lambda \in \mathfrak{h}^*$ and $\tau \in \widehat{K}$. Carrying the Hermitian dual over to $(\mathfrak{h}, K)$-modules we have $(\lambda, \tau)^h = (-\overline{\lambda}, \tau)$.

We can assume τ is the restriction of an element κ of $X^*(H)$. Then $(\mathfrak{h}, K)$-modules are parametrized by pairs (λ, κ) with $\lambda \in \mathfrak{h}^*, \kappa \in X^*(H)$, satisfying

$$\lambda - \theta^\vee \lambda = \kappa - \theta^\vee \kappa. \tag{5}$$

Furthermore (λ, κ) and (λ', κ') give the same character if and only if $\lambda' = \lambda$ and $\kappa' \in \kappa + (1 + \theta^\vee)X^*(H)$.

The differential of a character of $H(\mathbb{R})$ is contained in $i\mathfrak{t}_0^* + \mathfrak{a}^*$. So if (λ, κ) is a $(\mathfrak{g}, K)$-module, $\lambda + \overline{\lambda} \in \mathfrak{a}^*$. This implies

$$\theta^\vee(\lambda + \overline{\lambda}) = (\lambda + \overline{\lambda}). \tag{6}$$

In our language it is better to see this as follows. Since $X^*(H) \subset i\mathfrak{t}_0^* + \mathfrak{a}^*$, $\theta^\vee(\kappa + \overline{\kappa}) = (\kappa + \overline{\kappa})$. From (5) we have $\lambda - \theta^\vee \lambda = \kappa - \theta^\vee \kappa$, $\overline{\lambda} - \theta^\vee \overline{\lambda} = \overline{\kappa} - \theta^\vee \overline{\kappa}$, and adding these we see

$$(\lambda + \overline{\lambda}) - \theta^\vee(\lambda + \overline{\lambda}) = (\kappa + \overline{\kappa}) - \theta^\vee(\kappa + \overline{\kappa}) = 0.$$

Let $H^\vee$ be the dual torus, so $X^*(H) = X_*(H^\vee)$ etc., and we identify $\mathfrak{h}^\vee = \mathfrak{h}^*$, and carry $\theta^\vee$ over to $H^\vee$. Let $H^{\vee\Gamma} = H^\vee \rtimes \Gamma = \langle H^\vee, \delta^\vee \rangle$ be the L-group of H (where $\delta^{\vee 2} = 1$, and $\delta^\vee$ acts on $H^\vee$ by $\theta^\vee$). Suppose (y, λ) $(y \in H^{\vee\Gamma} \backslash H, \lambda \in X_*(H^\vee))$ satisfy $y^2 = \exp(2\pi i \lambda)$. This is equivalent to

$$\lambda - (\tau + \theta^\vee(\tau)) \in X^*(H) \tag{7}$$

where $y = \exp(2\pi i \tau)\delta^\vee$.

Then (y, λ) define a map $W_\mathbb{R} \to H^{\vee\Gamma}$ and hence a character χ of $H(\mathbb{R})$. The corresponding $(\mathfrak{h}, K)$-module is (λ, κ) where

$$\kappa = \lambda - (\tau + \theta^\vee(\tau)), \tag{8}$$

which is in $X^*(H)$ by (8).

2 Hermitian Dual of (y,λ)

Define $(y,\lambda)^h$ in the obvious way: this is the L-parameter corresponding to the Hermitian dual of the character defined by (y,λ).

Lemma 9

$$(y,\lambda)^h = (\exp(\pi i(\lambda-\overline{\lambda}))y^{-1}, -\overline{\lambda}). \tag{10}$$

Proof. Write $y=\exp(2\pi i\tau)\delta^\vee$. The $(\mathfrak{h},K)$-module defined by (y,λ) is (λ,κ) with $\kappa=\lambda-(\tau+\theta^\vee\tau)$. The Hermitian dual of this is $(-\overline{\lambda},\kappa)$.

Let $\mu=\frac{1}{2}(\lambda-\overline{\lambda})$, and note that

$$\exp(\pi i(\lambda-\overline{\lambda})y^{-1}) = \exp(2\pi i(\mu-\theta^\vee\tau))\delta^\vee. \tag{11}$$

Letting $\tau'=(\mu-\theta^\vee\tau)$ the parameter on the right hand side of (10) is $(\exp(2\pi i\tau')\delta^\vee,-\overline{\lambda})$, so we have to show

$$-\overline{\lambda}-(\tau'+\theta^\vee\tau') \in \kappa+(1+\theta^\vee)X^*(H) \tag{12}$$

So:
(13)

$$\begin{aligned}
-\overline{\lambda}-(\tau'+\theta^\vee\tau') &= -\overline{\lambda}-[(\mu-\theta^\vee\tau)+\theta^\vee(\mu-\theta^\vee\tau)]\\
&= -\overline{\lambda}-[(\mu+\theta^\vee\mu)-(\tau+\theta^\vee\tau)]\\
&= [(\tau+\theta^\vee\tau))-\lambda]+(\lambda-\overline{\lambda})-(\mu+\theta^\vee\mu)\\
&= -\kappa+(\lambda-\overline{\lambda})-(\mu+\theta^\vee\mu) \quad \text{(by (8))}\\
&= -\kappa+(\lambda-\overline{\lambda})-\frac{1}{2}[(\lambda-\overline{\lambda})+\theta^\vee(\lambda-\overline{\lambda})]\\
&\qquad \text{(by the definition of } \mu)\\
&= -\kappa+(\lambda-\overline{\lambda})-\frac{1}{2}[(\lambda+\overline{\lambda})-\theta^\vee(\lambda+\overline{\lambda})-2\overline{\lambda}+2\theta^\vee\lambda]
\end{aligned}$$

By (6) $(\lambda+\overline{\lambda})-\theta^\vee(\lambda+\overline{\lambda})=0$, so

$$\begin{aligned}
-\overline{\lambda}-(\tau'+\theta^\vee\tau') &= -\kappa+(\lambda-\overline{\lambda})+(\overline{\lambda}-\theta^\vee\lambda)\\
&= -\kappa+(\lambda-\theta^\vee\lambda)\\
&= -\kappa+(\kappa-\theta^\vee\kappa) \quad \text{(by (5))}\\
&= -\theta^\vee\kappa\\
&= \kappa-(1+\theta^\vee)\kappa \in \kappa+(1+\theta^\vee)X^*(H)
\end{aligned} \tag{14}$$

as required. ∎

We express this in terms of $\widetilde{\lambda}=\theta^\vee\overline{\lambda}$. Note that if $h=\exp(\pi i\widetilde{\lambda})$ then

$$\begin{aligned}h(\exp(\pi i(\lambda-\widetilde{\lambda}))y^{-1})h^{-1}&=\exp(\pi i(\lambda-\widetilde{\lambda})+\pi i(\widetilde{\lambda}-\theta^\vee\widetilde{\lambda}))y^{-1}\\&=\exp(\pi i(\lambda-\overline{\lambda}))y^{-1}.\end{aligned}\tag{15}$$

Therefore we can replace $\lambda-\overline{\lambda}$ with $\lambda-\widetilde{\lambda}$. We conclude

Lemma 16

$$\begin{aligned}(y,\lambda)^h&=(\exp(\pi i(\lambda-\widetilde{\lambda}))y^{-1},-\theta^\vee(\widetilde{\lambda}))\\&=(\exp(\pi i(\lambda-\widetilde{\lambda}))y^{-1},-Ad(y)\widetilde{\lambda})\\&=(\exp(\pi i(\lambda-\widetilde{\lambda}))y^{-1},-\overline{\lambda}).\end{aligned}\tag{17}$$

Given y, suppose $\exp(2\pi i\lambda)=y^2$. Then $\exp(2\pi i(\lambda-\widetilde{\lambda}))$ is independent of the choice of λ: any other choice is of the form $\lambda+\gamma$ with $\gamma\in X^*(H)$, and $(\lambda+\gamma)-\widetilde{(\lambda+\gamma)}=\lambda-\widetilde{\lambda}$ by (3).

Definition 18 *Given y, choose λ satisfying $\exp(2\pi i\lambda)=y^2$, and let*

$$y^h=\exp(\pi i(\lambda-\widetilde{\lambda}))y^{-1}.\tag{19}$$

We are primarily interested in the case of *real infinitesimal character*, which corresponds to $\widetilde{\lambda}=\lambda$. In this case

$$y^h=y^{-1}\tag{20(a)}$$

and

$$(y,\lambda)^h=(y^{-1},-\lambda).\tag{20(b)}$$

A closely related condition is that of integral infinitesimal character, i.e. $\langle\lambda,\alpha^\vee\rangle\in\mathbb{Z}$ for all roots; this implies the "semisimple part" of λ is real; in fact $\lambda-\widetilde{\lambda}$ is contained in the split part of the center of $\mathfrak{g}$, and $\exp(\pi i(\lambda-\widetilde{\lambda}))\in Z(G)$.

The opposite extreme is $\overline{\lambda}=-\lambda$, which corresponds to unitary characters, and tempered representations. In this case $\exp(\pi i(\lambda-\overline{\lambda}))=\exp(2\pi i\lambda)=y^2$, so

$$y^h=y\tag{21(a)}$$

and

(21)(b) $$(y,\lambda)^h = (y,\lambda).$$

which is reassuring.

3 Involution of KGB

Recall some notation from [1].

We let $\widetilde{\mathcal{X}} = \{\xi \in \mathrm{Norm}_{G^\Gamma\setminus G}(H) \,|\, \xi^2 \in Z(G)\}$, $\mathcal{X} = \widetilde{X}/H$. Then

$$\mathcal{X} \simeq \coprod_\xi K_\xi \setminus G/B$$

where the (disjoint) union is over strong real forms, i.e. G-conjugacy classes in $\widetilde{X}$.

If $\xi \in \widetilde{X}$ the involution θ_ξ is defined. If $x \in \mathcal{X}$ the involution $\theta_{x,H} = \theta_\xi|_H$ of H is well defined.

It is obvious the map $\xi \to \xi^{-1}$ on $\widetilde{X}$ descends to $\mathcal{X}$. This defines an automorphism of $\mathcal{X}$, which we write $x \to x^{-1}$.

The inner class of G gives an involution of the based root system (coming from the Cartan involution of the fundamental Cartan). This is an involution of the positive roots, and induces an automorphism of W. We define $W^\Gamma = W \rtimes \mathbb{Z}/2\mathbb{Z} = \langle W, \delta\rangle$ and $\mathcal{I}_W$ is the space of twisted involutions:

$$\begin{aligned}\mathcal{I}_W &= \{\tau \in W^\Gamma \setminus W \,|\, \tau^2 = 1\}\\ &\simeq \{w \in W \,|\, w\delta(w) = 1\}.\end{aligned}$$

There is a natural map $p : \mathcal{X} \to \mathcal{I}_W$.

Lemma 22

1. *$p(x^{-1}) = p(x)$*
2. *$\theta_{x^{-1},H} = \theta_{x,H}$*
3. *If $p(x) = w\delta$ then $w^{-1}(\theta_x(\Delta^+)) = \Delta^+$*

Proof. Part (1) is immediate from the defintions, and so is (2) since $\theta_{x,H}$ only depends on $p(x)$. For (3), if $\alpha > 0$ then $w^{-1}\theta_x(\alpha) = w^{-1}(w\delta(\alpha)) = \delta(\alpha) > 0$ since δ is an involution of the based root datum. $\square$

Fix $\xi \in \widetilde{X}$ with image $x \in \mathcal{X}$, and let

$$\mathcal{X}[x] = \{x' \in \mathcal{X} \mid x' \text{ is } G\text{-conjugate to } x\}$$

(the notion of G-conjugacy is well defined on $\mathcal{X}$). Then $\mathcal{X}[x] \simeq K_\xi\backslash G/B$.

It is obvious that

$$\mathcal{X}[x]^{-1} = \mathcal{X}[x^{-1}] \tag{23}$$

and $\mathcal{X}[x]^{-1} = \mathcal{X}[x]$ if and only if x^{-1} is G-conjugate to x.

Without loss of generality we can take $x \in H\delta$, and (after conjugating by H) assume that $x \in T$ (the identity component of H^δ). Then $x^{-1} = h^{-1}\delta$. Let W_i be the Weyl group of the δ-imaginary roots.

Lemma 24 *Suppose $\xi = h\delta$ and $\delta(h) = h$, and let x be the image of ξ in $\mathcal{X}$.*

1. *$\mathcal{X}[x]^{-1} = \mathcal{X}[x]$ if and only if h^{-1} is W_i-conjugate to h.*
2. *Write $h = \exp(2\pi i\tau^\vee)$ with $\tau^\vee \in X_*(H) \otimes \mathbb{C} = \mathfrak{h}$. Then $\mathcal{X}[x]^{-1} = \mathcal{X}[x]$ if and only if $\tau + w\tau \in X_*(H)$ for some $w \in W_i$.*

See the proof of [1, Proposition 2.12].

Note that there is no a priori reason for this to hold. For example if G is a torus and $\delta = 1$ this holds if and only if h has order 2.

Example 25 The worst failure of $x^{-1} \sim x$ occurs in the the compact inner class of $G = SL(n, \mathbb{C})$ (with corresponding real groups $SU(p, q)$). Suppose $p + q = n$, $\alpha^n = (-1)^q$,

$$h = \mathrm{diag}(\overbrace{\alpha, \dots, \alpha}^{p}, \overbrace{-\alpha, \dots, -\alpha}^{q})$$

and $x = h\delta$. The corresponding real form is $SU(p, q)$. If $p \neq q$ then h is conjugate to h^{-1} if and only if $\alpha = \pm 1$; if $p = q$ we allow $\alpha = \pm 1, \pm i$.

If p or q is even then $\mathcal{X}[x]^{-1} = \mathcal{X}[x]$ if

$$h = \mathrm{diag}(1, \dots, 1, -1, \dots, -1); \tag{26}$$

these give the groups $SU(p,q)$ with p or q even. This also holds if

$$h = \mathrm{diag}(\overbrace{i,\dots,i}^{p},\overbrace{-i,\dots,-i}^{p}). \tag{27}$$

which gives $SU(p,p)$ with p odd.

I think this never happens in types B/C, but does in type D. It cannot happen in types E_8, F_4, G_2 (which are always adjoint), so the only other places this could arise are E_6 and E_7.

4 Involution of $\mathcal{Z}$

Fix $G, G^\vee$, let $\mathcal{Z} = \mathcal{X} \times \mathcal{X}^\vee$ the parameter space of representations. By the previous discussion the map $(x,y) \to (x, y^{-1})$ is an involution of $\mathcal{Z}$.

Recall that W acts on $\mathcal{X}, \mathcal{X}^\vee$ and $\mathcal{Z}$.

Definition 28 *Supppose $(x,y) \in \mathcal{Z}$. Write $p(x) = w_x\delta$, so w_x is the last entry in the output of* `kgb` *for x. Let*

$$(x,y)^h = (w_x^{-1} x w_x, w_x^{-1} y^{-1} w_x). \tag{29}$$

Practically speaking we can think of this as

$$(x,y)^h = w_x^{-1} \times (x, y^{-1}) \tag{30}$$

where we compute $w_x^{-1}\times$ using the output of block.

This is an involution of $\mathcal{Z}$.

5 Hermitian dual in (x,y) parameters

Recall (x,y) gives a *translation family* of representations. Here is how to pin down the infinitesimal character. Assume $\exp(2\pi i\lambda) = y^2$. We assume λ is integral, so $y^2 \in Z(G)$. We also assume λ is real, i.e. $\widetilde{\lambda} = \lambda$; given integrality this is only a condition on the split part of the center (see the end of Section 2).

We have fixed a set of positive roots. If $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all α then the parameter (x, y, λ) is defined, and defines a $(\mathfrak{g}, K_\xi)$-module (where ξ lies over x) with infinitesimal character λ.

Proposition 31 *Suppose π corresponds to parameter (x, y, λ). Write $p(x) = w_x\delta$. Then π^h is given by the parameter*

$$(x, y, \lambda)^h = (w_x^{-1}xw_x, w_x^{-1}y^{-1}w_x, -w_x\overline{\lambda}). \tag{32}$$

Note that π and π^h have the same infinitesimal character if and only if $-\overline{\lambda}$ is W-conjugate to λ. Assuming this holds (x, y, λ) and $(x, y, \lambda)^h$ are in the same block if and only if y^h is conjugate to y.

Corollary 33 *Assume λ is real, $-\overline{\lambda}$ is conjugate to λ, and y^{-1} is conjugate to y. Then $\gamma \to \gamma^h$ is an automorphism of the block (with infinitesimal character λ). It is given by:*

$$\begin{aligned}(x, y)^h &= (w_x^{-1}xw_x, w_x^{-1}y^{-1}w_x) \\ &= w_x^{-1} \times (x, y^{-1}).\end{aligned} \tag{34}$$

Here $p(x) = w_x\delta$ (given by the last entry in the output of `kgb`*) and $w_x^{-1} \times (x, y^{-1})$ is the cross action, which can be computed via the output of* `block`*.*

Remark 35 Once we have set things up for nonintegral infinitesimal character, the corresponding result in general should be something like this.

Let $W(\lambda)$ be the integral root system of λ. Choose $w \in W(\lambda)$ so that

$$\mathrm{Re}\langle -w\overline{\lambda}, \alpha^\vee\rangle \geq 0 \quad \text{(for all } \alpha). \tag{36}$$

Then π^h is given by parameter

$$(x, y, \lambda)^h = (w^{-1}xw, w^{-1}y^hw, -w\overline{\lambda}) \tag{37}$$

For example if $\overline{\lambda} = -\lambda$ then $y^h = y, w = 1$ and

$$(x, y, \lambda)^h = (x, y, \lambda). \tag{38}$$

If λ is real and integral then $w = w_x$, $y^h = y^{-1}$ and this agrees with Proposition 31.

Example 39 Let $G = GL(3, \mathbb{R})$. Here is the large block:

```
0(0,5):  0  0  [C+,C+]  2  1   (*,*)  (*,*)
1(1,4):  1  0  [i2,C-]  1  0   (3,4)  (*,*)   2,1
2(2,3):  1  0  [C-,i2]  0  2   (*,*)  (3,5)   1,2
3(3,0):  2  1  [r2,r2]  4  5   (1,*)  (2,*)   1,2,1
4(3,1):  2  1  [r2,rn]  3  4   (1,*)  (*,*)   1,2,1
5(3,2):  2  1  [rn,r2]  5  3   (*,*)  (2,*)   1,2,1
```

and here is kgb:

```
Name an output file (return for stdout, ? to abandon):
0:  0  0  [C,C]   2  1    *  *
1:  1  0  [n,C]   1  0    3  *  2,1
2:  1  0  [C,n]   0  2    *  3  1,2
3:  2  1  [r,r]   3  3    *  *  1,2,1
```

In this case every kgb element for $G^\vee$ has order 2. Let $\lambda = \rho$ Therefore

$$
\begin{aligned}
&\#0 = (0,5) \to (0,5)^h = (0,5) = \#0\\
&\#1 = (1,4) \to (1,4)^h = 1 \times 2 \times (1,4) = (2,3) = \#2\\
&\#2 = (2,3) \to (2,3)^h = 2 \times 1 \times (2,3) = (1,4) = \#1\\
&\#3 = (3,0) \to (3,0)^h = 1 \times 2 \times 1 \times (3,0) = (3,0) = \#3\\
&\#4 = (3,1) \to (3,1)^h = 1 \times 2 \times 1 \times (3,1) = (3,2) = \#5\\
&\#5 = (3,2) \to (3,2)^h = 1 \times 2 \times 1 \times (3,2) = (3,1) = \#4
\end{aligned} \tag{40}
$$

We check this example another way in the next Section.

6 Examples: Classical Groups

It is easy to compute the Hermitian dual in terms of Barbasch's parameters for classical groups. The `parameters` software on the web site (software/helpers) gives block output in these parameters. See the help file for the software for more details.

For example the parameter

$$\gamma = (6+, 5-, \underline{4\ 3}, 2, 1)$$

for $Sp(12,\mathbb{R})$ corresponds toa a representation π which is induced from $M = \mathbb{R}^{x2} \times GL(2,\mathbb{R}) \times Sp(4,\mathbb{R})$. The only thing which isn't fairly obvious is the representation of $GL(2,\mathbb{R})$, see below. In any event the Hermitian dual π^h of π has parameter

$$\gamma^h = (-6+, -5-, \underline{-3\ \ -4}, 2, 1)$$

which is equivalent to γ, i.e. this representation is Hermitian.

$GL(2,\mathbb{R})$ factors:

A term $\underline{a\ b}$ or $\overline{a\ b}$ means the torus contains a copy of $\mathbb{C}^\times$:

$(\underline{\mathrm{a\ b}})$ means $e_1 - e_2$ is imaginary, $e_1 + e_2$ is real (BCD). In particular $a - b \in \mathbb{Z}$.

$(\overline{a\ b})$ means $e_1 - e_2$ is real, $e_1 + e_2$ is real (BCD). In particular $a + b \in \mathbb{Z}$.

Note that

$$(41)\qquad (\underline{a\ b}) = (\overline{a\ \ -b}).$$

Note that since the imaginary reflection $s_{e_1-e_2}$ is in the Weyl group for $GL(2,\mathbb{R})$, we can always replace $(\underline{a\ b})$ with $(\underline{b\ a})$ and get an equivalent representation. In types BCD, the same holds for the real reflection $(\underline{a\ b}) \to (\underline{-b\ \ -a})$.

Similarly the real reflection $(\overline{a\ b}) \to (\overline{b\ a})$ is always allowed, and the imaginary reflection $(\overline{a\ b}) \to (\overline{-b\ \ -a})$ in types BCD.

Write $[k,\nu]$ for the character of $\mathbb{C}^*$: $re^{i\theta} \to r^\nu e^{ik\theta}$.

Here is the dictionary going between $(\underline{a\ b})$ or $(\overline{a\ b})$ and $[k,\nu]$:

$$(42)\qquad \begin{aligned} (\underline{a\ b}) &\to [a-b, a+b] \\ (\underline{\tfrac{1}{2}(k+\nu)\ \ \tfrac{1}{2}(-k+\nu)}) &\leftarrow [k,\nu] \\ (\overline{a\ b}) &\to [a+b, a-b] \\ (\overline{\tfrac{1}{2}(k+\nu)\ \ \tfrac{1}{2}(k-\nu)}) &\leftarrow [k,\nu] \end{aligned}$$

Now the Hermitian dual is

$$(43)\qquad [k,\nu]^h = [k, -\overline{\nu}].$$

Chasing this around we compute

$$(44)\qquad \begin{aligned} (\underline{a\ b})^h &= ([-\mathrm{Re}(b) + i\mathrm{Im}(a)]\_[-\mathrm{Re}(a) + i\mathrm{Im}(b)]) \\ (\overline{a\ b})^h &= ([\mathrm{Re}(b) + i\mathrm{Im}(a)]\hat{}[\mathrm{Re}(a) + i\mathrm{Im}(b)]) \end{aligned}$$

The infinitesimal character is real if $a, b \in \mathbb{R}$, in which case it is much easier:

$$(45)\qquad \begin{aligned} (\underline{a\ b})^h &= (\underline{-b\ \ -a}) = -(\underline{b\ a}) \\ (\overline{a\ b})^h &= (\overline{b\ a}). \end{aligned}$$

Example 46 We illustrate the fact that even if π and π^h have the same infinitesimal character, π cannot be Hermitian if y^{-1} is not conjugate to y.

Let $G = PSL(4,\mathbb{C})$, $G(\mathbb{R}) = PSL(4,\mathbb{R})$, the split real form of $PSL(4,\mathbb{C})$. It is easiest to think of this group as $GL(4,\mathbb{R})/\mathbb{R}^\times$.

There are four compact strong real forms of $G^\vee = SL(4,\mathbb{C})$, given by elements of the center $y = \pm I, \pm iI$. These correspond to four irreducible principal series representations of $G(\mathbb{R})$. See the Remark below.

We assume λ is real and $-\lambda$ is W-conjugate to λ, i.e. $-w_0\lambda = \lambda$ where w_0 is the long element of the Weyl group.

In terms of (x,y) note that $w_x = w_0$, $-w_0\lambda = \lambda$, and

$$\begin{aligned}(x,y,\lambda)^h &= (w_0^{-1}xw_0, w_0^{-1}yw_0, -w_0\lambda)\\ &= (x, y^{-1}, \lambda).\end{aligned} \tag{47}$$

Thus the representation corresponding to (x,y,λ) is Hermitian if and only if $y^{-1} = y$.

Suppose $y = \pm I$. We can take the infinitesimal character to be all integers, for example $\lambda = (2,1,-1,-2)$. If $y = I$ take

$$\gamma_I = (2+, 1-, -1-, -2+). \tag{48}$$

(To be precise this is a representation of $GL(4,\mathbb{R})$, in Barbasch's notation, which factors to $G(\mathbb{R})$.) For $\gamma = -I$ we have

$$\gamma_I = (2-, 1+, -1+, -2-). \tag{49}$$

It is easy to see $\pi(\gamma_{\pm I})$ are Hermitian:

$$\begin{aligned}\gamma_I^h &= (2+, 1-, -1-, -2+)^h\\ &= (-2+, -1-, 1-, 2+)\\ &= (2+, 1-, -1-, -2+) = \gamma_I^h.\end{aligned} \tag{50}$$

corresponding to the fact that in this case y^{-1} is conjugate to (in fact equal to) y.

Now suppose $y = +iI$, so $y^2 = -I$, and the corresponding infinitesimal character is in $\rho + X^*(H)$. We can take

$$\gamma_{iI} = (3/2+, 1/2-, -1/2+, -3/2-). \tag{51}$$

On the other hand if $y = -iI$ then the infinitesimal character is the same, and we can take

$$\gamma_{-iI} = (3/2-, 1/2, -1/2-, -3/2+). \tag{52}$$

Even though $\pi(\gamma_{iI})$ and $\pi(\gamma_{-iI})$ have the same infinitesimal character, they are not Hermitian; $\pi(\gamma_{iI}) = \pi(\gamma_{-iI})$. This is easy to see:

$$\begin{aligned} \gamma_{iI}^h &= (3/2+, 1/2-, -1/2+, -3/2-)^h \\ &= (-3/2+, -1/2-, 1/2+, 3/2-) \\ &= (3/2-, 1/2+, -1/2-, -3/2+) = \gamma_{-iI} \end{aligned} \tag{53}$$

This corresponds to the fact that $y_{iI}^{-1} = y_{-iI}$ is not conjugate to y_{iI}.

Remark 54 Note that there are 4 compact strong real forms of $SL(4,\mathbb{C})$, corresponding to the 4 singleton blocks of $PSL(4,\mathbb{R})$ (up to tranlslation), say at infinitesimal character ρ and $\lambda' = (2,1,-1,-2)$. The two blocks at ρ differ by tensoring with sgn, as do the two at λ'. The two blocks at λ' are Hermitian, while the two blocks at ρ are each other's Hermitian duals.

Also note that `atlas` only sees two of the strong real forms, say $y = I$ and $y = iI$. The strong real forms $\pm I$ are equivalent in the sense of the reduced parameter space, as are $\pm iI$. The example shows that some information is lost when passing to the reduced parameter space.

```
main: strongreal
(weak) real forms are:
0: su(4).u(1)
1: su(3,1).u(1)
2: su(2,2).u(1)
enter your choice: 0
there is a unique conjugacy class of Cartan subgroups
Name an output file (return for stdout, ? to abandon):

real form #2: [0,1,2,8,9,10] (6)
real form #0: [3] (1)
real form #1: [4,6,7,13] (4)
real form #1: [5,12,14,15] (4)
real form #0: [11] (1)
```

Example 55 We do Example 39 in these terms. Here $G = GL(3,\mathbb{R})$ and $G^\vee = U(2,1)$.

```
Command:parameters -t A -b inputFiles/blockGL3 -s 3=(1+,0+,-1+)
G=GL(3,R) (based on the block file inputFiles/blockGL3)

Computed Parameters:
Barbasch: parameters in Barbasch's notation
Action: how the parameter was obtained
Atlas: parameter from atlas block file inputFiles/blockGL3

Barbasch        Action      Atlas
(1_-1,0-)       2x1        0(0,5):  0  0  [C+,C+]  2  1   (*,*)  (*,*)
(1_0,-1+)       1^3        1(1,4):  1  0  [i2,C-]  1  0   (3,4)  (*,*)   2,1
(-1_0,1+)       2^3        2(2,3):  1  0  [C-,i2]  0  2   (*,*)  (3,5)   1,2
(1+,0+,-1+)     ***        3(3,0):  2  1  [r2,r2]  4  5   (1,*)  (2,*)   1,2,1
(1-,0-,-1+)     1x3        4(3,1):  2  1  [r2,rn]  3  4   (1,*)  (*,*)   1,2,1
(1+,0-,-1-)     2x3        5(3,2):  2  1  [rn,r2]  5  3   (*,*)  (2,*)   1,2,1
```

In this example representations #0,#3 are Hermitian, and the Hermitian dual operations interchanges #1,#2, and also #4,#5. Fore example

$$\#1 = (\underline{1\ 0}, -1+) \to (\underline{1\ 0}, -1+)^h = (\underline{0\ -1}, 1+) = (\underline{-1\ 0}, 1+) = \#2$$

and

$$\#4 = (1-, 0-, -1+) \to (1-, 0-, -1+)^h = (-1-, 0-, 1+) = (1+, 0-, -1-) = \#5.$$

This agrees with Example 39.

References

[1] Jeffrey Adams and F. du Cloux. Algorithms for representation theory of real reductive groups. preprint.