Notes on Doubly Extended Groups

Jeffrey Adams

March 8, 2009

These are notes on the extended group formalism, used in computing Hermitian representations. Also see *Notes on the Hermitian Dual* [2].

1 Setup

The basic setup is a complex group G, with a fixed pinning $H_0, B_0, \{X_\alpha\}$. Recall an involution θ of G is *distinguished* if it fixes the pinning; θ is the Cartan involution of the "maximally compact" form in its inner class.

The usual starting point is a distinguished involution τ, and $G^\Gamma = \langle G, \delta\rangle$ where $\delta^2 = 1$ and $\delta g \delta^{-1} = \tau(g)$.

Definition 1 *Fix two commuting, distinguished involutions τ, μ. The doubly extended group is $G^{\Gamma\dagger} = \langle G, \delta, \epsilon\rangle$ where $\delta^2 = \epsilon^2 = 1, \delta\epsilon = \epsilon\delta$, and*

$$\delta g \delta^{-1} = \tau(g), \quad \epsilon g \epsilon^{-1} = \mu(g). \tag{2}$$

Also let

$$G^\Gamma = \langle G, \delta\rangle, \quad G^\dagger = \langle G, \epsilon\rangle. \tag{3}$$

We think of τ as given an inner class of real forms of G; $G^\Gamma = \langle G, \delta\rangle$ is the the usual extended group. By a *strong real form* for G we mean with respect to G^Γ, i.e. in the inner class of τ. The involution μ is secondary.

An important special case is $\mu = \tau$, which governs Hermitian representations of strong real forms G (in the inner class of τ). Often on the dual side $\mu \neq \tau$.

2 KGB

Define the spaces $\widetilde{\mathcal{X}}$ and $\mathcal{X}$ for G^Γ. That is

$$\widetilde{\mathcal{X}} = \{\xi \in \mathrm{Norm}_{G^\Gamma \backslash G}(H_0) \mid \xi^2 \in Z(G)\} \tag{4}$$

and

$$\mathcal{X} = \widetilde{\mathcal{X}}/H_0 \tag{5}$$

(the quotient of $\widetilde{\mathcal{X}}$ by the conjugation action of H_0). Write $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ for the projection map.

Recall we fix a set $\{\xi_i \mid i \in \mathcal{I}\} \subset \widetilde{\mathcal{X}}$ so that every element of $\widetilde{\mathcal{X}}$ is G-conjugate to precisely one ξ_i. For $i \in \mathcal{I}$ we let $\theta_i = \mathrm{int}(\xi_i)$ and $K_i = \mathrm{Cent}_G(\xi_i) = G^{\theta_i}$. Then

$$\mathcal{X} \longleftrightarrow \coprod_{i \in \mathcal{I}} K_i \setminus G/B_0. \tag{6}$$

We make repeated use of the form of this bijection.

Lemma 7 *Suppose $x \in \mathcal{X}$. Choose $\xi \in \widetilde{\mathcal{X}}$ mapping to x, and $i \in \mathcal{I}, g \in G$ so that $g\xi_i g^{-1} = \xi$. Then the image of x under the bijection* (6) *is the K_i-orbit of $g^{-1}B_0 g$.*

Note that the choices of ξ and g do not affect i or the K_i-orbit $g^{-1}B_0 g$. These choices amount to replacing g by hgk with $h \in H_0$ and $k \in K_i$, and note that $(hgk)^{-1}B_0(hgk) = k^{-1}(g^{-1}B_0 g)k$.

3 Action of μ on representations

In the setting of Section 1 suppose θ is an involution in the inner class of τ. and (π, V) is a $(\mathfrak{g}, K)$ module. Then we obtain a $(\mathfrak{g}, \mu(K))$ module by twisting by μ as usual: $\pi^\mu(X)(v) = \pi(\mu^{-1}(X))(v)$ and $\pi^\mu(g)(v) = \pi(\mu^{-1}(g))(v)$ $(X \in \mathfrak{g}, g \in \mu(K))$.

In the atlas setting, suppose $x \in \mathcal{X}$ and $\xi, \xi' \in \widetilde{\mathcal{X}}$ are inverse images. Then we canonically identify $(\mathfrak{g}, K_\xi)$ and $(\mathfrak{g}, K_{\xi'})$-modules. If a $(\mathfrak{g}, K_\xi)$-module is given by a $(\mathfrak{h}_0, H_0^{\theta_x})$-module χ, the corresponding $(\mathfrak{g}, K_{\xi'})$-modules is also given by χ.

Suppose $x \in \mathcal{X}$ satisfies $\mu(x) = x$, and chooseo $\xi \in \widetilde{\mathcal{X}}$ lying over x. This allows us to define an action of μ on $(\mathfrak{g}, K_\xi)$-modules. If we can choose $\mu(\xi)$ then $\mu(K_\xi) = K_\xi$ and this action is obvious, but in general $\mu(x) = x$ does not imply we can choose ξ such that $\mu(\xi) = \xi$. See Remark 33.

So fix $x \in \mathcal{X}$ satisfying $\mu(x) = x$. We're interested in parametrizing pairs $((\pi, V), \psi)$ where (π, V) is an irreducible $(\mathfrak{g}, K)$-module such that (π^μ, V) is isomorphic to (π, V), and $\psi : V \to V$ intertwines π^μ and π. Choose an inverse image ξ of x in $\widetilde{\mathcal{X}}$, and let $K_\xi = \mathrm{Cent}_G(\xi)$ as usual.

Definition 8 *Let* $K_\xi^\dagger = Cent_{G^\dagger}(\xi)$.

Lemma 9 *Fix* ξ *and let* $K = K_\xi$, $K^\dagger = K_\xi^\dagger$. *Suppose* $(\pi^\dagger, V)$ *is an irreducible* $(\mathfrak{g}, K^\dagger)$*-module.*

Suppose $\pi^\dagger|_{(\mathfrak{g},K)}$ *is irreducible, and write this restriction as* (π, V). *Then* $\pi^\dagger(\epsilon) : V \to V$ *intertwines* (π, V) *and* (π^μ, V).

Suppose $\pi^\dagger$ *is reducible. Then this restriction can be written* $(\pi_1, V_1) \oplus (\pi_2, V_2)$, *and* $\pi^\dagger(\epsilon) : V_1 \to V_2$ *is an intertwining operator* $(\pi_1^\mu, V_1) \simeq (\pi_2, V_2)$.

So we consider irreducible $(\mathfrak{g}, K^\dagger)$-modules, with an emphasis on those whose restriction to $(\mathfrak{g}, K)$ is irreducible.

4 Automorphism of $\mathcal{X}$ and $\mathcal{Z}$

Fix $G^{\Gamma,\dagger}$ as in Section 1.

Recall if $\xi \in \widetilde{\mathcal{X}}$ then the automorphism θ_ξ is defined. Furthermore if $x \in \mathcal{X}$ choose $\xi \in \widetilde{\mathcal{X}}$ lying over x and define the automorphism θ_{x,H_0} of H_0 by $\theta_{x,H_0}(h) = \theta_\xi(h)$ This is well defined, independent of the choice of ξ. We write $H_0^{\theta_x}$ for the fixed points (strictly speaking we should write $H_0^{\theta_{x,H_0}}$ or $H_0^{\theta_\xi}$).

Now define $G^\vee, \tau^\vee, G^{\vee\Gamma}, H_0^\vee$ and $\mathcal{X}^\vee$ as usual. In particular $X^*(H_0^\vee) = X_*(H_0)$ and $X_*(H_0^\vee) = X^*(H_0)$. Suppose ϕ is an automorphism of $H_0 = X_*(H_0) \otimes \mathbb{C}^*$. Then ϕ^t is the automorphism of $X^*(H_0)$ given by the pairing $X^*(H_0) \times X_*(H_0) \to \mathbb{Z}$, and this also defines an automorphism ϕ^t of $X_*(H_0^\vee), X^*(H_0^\vee)$ and $H_0^\vee$.

Recall

$$\mathcal{Z} = \{(x, y) \in \mathcal{X} \times \mathcal{X}^\vee \,|\, \theta_{y,H^\vee} = -\theta_{x,H}^t\} \tag{10}$$

Recall the distinguished automorphism μ of G induces an automorphism, also denoted μ, of the based root datum $D_b(G) = (X^*(H_0), \Pi, X_*(H_0), \Pi^\vee)$.

Definition 11 *Let μ^t be the automorphism of $D_b(G^\vee)$ obtained from μ by reversing the order of the factors, as well as the corresponding automorphism of $G^\vee$.*

Thus the action of μ^t on $H_0^\vee$ is described above. Furthermore if α is a simple root of H_0 in G, then $\alpha^\vee$ is a simple root of $H_0^\vee$ in $G^\vee$, and $\mu^t(X_{\alpha^\vee}) = X_{\mu(\alpha)^\vee}$.

Lemma 12 *The automorphism μ of G restricts to an automorphism of $\widetilde{\mathcal{X}}$, and this factors to an automorphism (also denoted μ) of $\mathcal{X}$.*

Proof. Since μ normalizes H_0 it is obvious that it acts on $\widetilde{\mathcal{X}}$. If $x \in X$ choose an inverse image $\xi \in \widetilde{\mathcal{X}}$, and define $\mu(x) = p(\mu(\xi))$. If ξ' is another choice, then $\xi' = h\xi h^{-1}$ for $h \in H_0$, and $p(\xi') = p(h\xi h^{-1}) = p(h)p(\xi)p(h^{-1}) = p(\xi)$. □

We also apply this to μ^t to get an automorphism of $\mathcal{X}^\vee$.

Proposition 13 *For $(x, y) \in \mathcal{Z}$ define $\mu(x, y) = (\mu(x), \mu^t(y))$. This is an automorphism of $\mathcal{Z}$.*

Proof. The only thing we need to check is the transpose relationship in (10), which is immediate. □

5 Cayley Transforms and Cross Actions

Fix $G^{\Gamma,\dagger}$ as in Section 1.

We consider how the automorphism μ of $\mathcal{Z}$ of Proposition 13 behaves with respect to Cayley transforms and cross actions, and how to obtain new μ-fixed parameters from old ones.

Recall the Tits group is generated by $\{\sigma_\alpha \mid \alpha \in \Pi\}$ (Π is the simple roots) and the elements of order 2 in H. Since μ is distinguished it induces an automorphism of the Tits group, and for all $\alpha \in \Pi$, $\mu(\sigma_\alpha) = \sigma_{\mu(\alpha)}$.

Lemma 14 *Fix $\gamma \in \mathcal{Z}$.*

1. *For all α $\mu(s_\alpha \times \gamma) = s_{\mu(\alpha)} \times \mu(\gamma)$.*

2. *If α is γ-imaginary then $\mu(\alpha)$ is $\mu(\gamma)$-imaginary and $\mu(c^\alpha(\gamma)) = c^{\mu(\alpha)}\mu(\gamma)$ (as sets).*

3. *If α is γ-real then $\mu(\alpha)$ is $\mu(\gamma)$-real and $\mu(c_\alpha(\gamma)) = c_{\mu(\alpha)}\mu(\gamma)$ (as sets).*

This is routine.

Lemma 15 *Suppose $\mu(\gamma) = \gamma$ and $\mu(\alpha) = \alpha$.*

1. *If $\mu(\alpha) = \alpha$ then $\mu(s_\alpha \times \gamma) = s_\alpha \times \gamma$.*

2. *If α is γ-imaginary and $\gamma' \in c^\alpha(\gamma)$ then $\mu(\gamma') = \gamma'$.*

3. *If α is γ-real and $\gamma' \in c_\alpha(\gamma)$ then $\mu(\gamma') = \gamma'$.*

Proof. Part (1) follows easily from the previous Lemma, as do (2) and (3) if the Cayley transforms are single valued.

Suppose α is γ-real. Write $\gamma = (x, y)$ and choose an inverse image $\xi \in \widetilde{\mathcal{X}}$ of x. There is a dangerous bend here: it is *not* necessarily the case that $c^\alpha(x)$ is the image of $\{\sigma_\alpha\xi, m_\alpha\sigma_\alpha\xi\}$ in $\mathcal{X}$ where σ_α is the Tits group element. However we can *choose* ξ so that this holds. That is (cf. [4, Lemma 14.6]) for any ξ there exists $\widetilde{\sigma}_\alpha$ such that $\widetilde{\sigma}_\alpha\xi$ is conjugate to ξ. We are free to vary ξ by conjugation by $h \in H_\alpha \subset G_\alpha$ (G_α is locally isomorphic to $SL(2)$), which amounts to varying $\widetilde{\sigma}_\alpha$) by conjugation. Since σ_α acts on $H_\alpha \simeq \mathbb{C}^*$ by $z \to z^{-1}$, we can replace $\widetilde{\sigma}_\alpha$ with the Tits element σ_α.

Now with this choice of ξ we have γ_1 (say) is the image of $(\sigma_\alpha\xi, \sigma_{\alpha^\vee}\xi^\vee)$ where $\xi^\vee$ is any choice of inverse image of y (cf. [4, Lemma 14.2]), and $\gamma_2 = (m_\alpha\sigma_\alpha\xi, \sigma_{\alpha^\vee}\xi^\vee)$. Now it is clear that $\mu(\gamma_i) = \gamma_i$ ($i = 1, 2$). This proves (3), and (2) is similar.

It is a little more work to get a new μ-fixed parameter if $\mu(\alpha) \neq \alpha$. First consider the case when $\alpha, \mu(\alpha)$ are orthogonal.

Lemma 16 *Suppose $\mu(\gamma) = \gamma$. Let $\beta = \mu(\gamma)$ and suppose $\langle\beta, \alpha^\vee\rangle = 0$. Let*

1. *$\mu(s_\beta s_\alpha(\gamma)) = s_\beta s_\alpha(\gamma)$.*

2. *Suppose α is imaginary and $\gamma' \in c^\beta(c^\alpha(\gamma))$. Then $\mu(\gamma') = \gamma'$.*

3. *Suppose α is real and $\gamma' \in c_\beta(c_\alpha(\gamma))$. Then $\mu(\gamma') = \gamma'$.*

Again (1) is immediate, and (2) and (3) require the same type of argument as in the previous Lemma.

Finally the case where $\alpha \neq \mu(\alpha)$, $\langle \mu(\alpha), \alpha^\vee \rangle \neq 0$ is a little harder. This only occurs for the middle two roots in type A_{2n}. □

Conjecture 17 *Suppose $\mu(\gamma) = \gamma$, $\beta = \mu(\alpha) \neq \alpha$ and $\langle \beta, \alpha^\vee \rangle \neq 0$. Then α, β are of the same type (complex, real or imaginary) with respect to γ.*

1. *Suppose α, β are complex and α (and therefore also β) is of type $C+$ with respect to γ. Then β is non-compact imaginary type 2 with respect to $s_\alpha \times \gamma$, and exactly one of the two constituents of $c^\beta(s_\alpha\gamma)$ is fixed by μ.*

2. *Suppose α, β are complex and α (and therefore also β) is of type $C-$ with respect to γ. Then β is a real, non-parity, type 2 root with respect to $s_\alpha \times \gamma$, and exactly one of the two constituens of $c_\beta(s_\alpha\gamma)$ is fixed by μ.*

sketch of proof. This holds for $SL(3, \mathbb{R})$ and dually $SU(2, 1)$. The general case should reduce to these.

```
empty: type
Lie type: A2 sc s
main: block
there is a unique real form: sl(3,R)
possible (weak) dual real forms are:
0: su(3)
1: su(2,1)
enter your choice: 1
Name an output file (return for stdout, ? to abandon):
0(0,5):  0  0  [C+,C+]  2  1   (*,*)  (*,*)
1(1,4):  1  0  [i2,C-]  1  0   (3,4)  (*,*)   2,1
2(2,3):  1  0  [C-,i2]  0  2   (*,*)  (3,5)   1,2
3(3,0):  2  1  [r2,r2]  4  5   (1,*)  (2,*)   1,2,1
4(3,1):  2  1  [r2,rn]  3  4   (1,*)  (*,*)   1,2,1
5(3,2):  2  1  [rn,r2]  5  3   (*,*)  (2,*)   1,2,1
block: dualblock
Name an output file (return for stdout, ? to abandon):
```

```
0(0,3):  0  0  [i1,i1]  1  2   (4,*)  (3,*)
1(1,3):  0  0  [i1,ic]  0  1   (4,*)  (*,*)
2(2,3):  0  0  [ic,i1]  2  0   (*,*)  (3,*)
3(3,2):  1  1  [C+,r1]  5  3   (*,*)  (0,2)   2
4(4,1):  1  1  [r1,C+]  4  5   (0,1)  (*,*)   1
5(5,0):  2  1  [C-,C-]  3  4   (*,*)  (*,*)   1,2,1
```

□

6 Parameters

This section is a bit painful, but the main result is more or less obvious: Proposition 29 shows how the action of μ on $(\mathfrak{g}, K_0)$-modules carries over to parameters.

As in Section 1 we're interested in $(\mathfrak{g}, K)$-modules for some K. So fix a strong involution ξ_0 in our given set of representatives, and let $\theta_0 = \theta_{\xi_0}$, $K_0 = \mathrm{Cent}_G(\xi_0)$. Also let $x_0 = p(\xi_0) \in \mathcal{X}$, and we assume

$$\mu(x_0) = x_0. \tag{18}$$

See Remark 33. Assume $\mu(\xi_0) = \xi_0$. We're interested in how μ acts on $(\mathfrak{g}, K_0)$-modules.

Remark 19 At least if $\mu = \tau$, $\mu(K_0) = K_0$ implies we can choose ξ_0 satisfying $\mu(\xi_0) = \xi_0$. To see this, recall we can assume ξ_0 is the fiber of δ, i.e. $\xi_0 = h\delta$ with $h \in H_0$. Conjugating by $t \in H_0$ gives $t\tau(t^{-1})h\delta$. Recall $H_0 = T_0A_0$ where T_0 (resp. A_0) is the identity component of $H_0\tau$ (resp. $H_0^{-\tau}$). Also $A_0 = \{t\tau(t^{-1}) \,|\, t \in H_0\}$. Therefore by choosing t we can assume $h \in T_0$. Then $\tau(h\delta) = h\delta$.

See Remark 33.

If we need more general cases I'll have to revisit this.

Recall a $(\mathfrak{g}, K_0)$-modules is given by a local system on a K_0-orbit on the variety of Borel subgroups. Fix a Borel subgroup B. Then

$$\mathrm{Stab}_{K_0}(B) = H^{\theta_0}U$$

where U is a connected unipotent group and H is a θ_0-stable Cartan subgroup of B. Then a $(\mathfrak{g}, K_0)$-module is determined by an $(\mathfrak{h}, \widetilde{H}^{\theta_0})$-module. Here $\widetilde{H}^{\theta_0}$ is the ρ-cover of H^{θ_0}.

Definition 20 *A complete parameter for G is a triple (x, y, λ_0) where*

1. $(x, y) \in \mathcal{Z}$
2. $\lambda_0 \in \mathfrak{h}_0 = X^*(H_0) \otimes \mathbb{C}$
3. $\langle \lambda_0, \alpha^\vee \rangle > 0$ *for all simple roots* α,
4. $\exp(2\pi i \lambda_0) = y^2$.

In (4) we identify λ_0 with an element of $X_*(H_0^\vee) \otimes \mathbb{C} = \mathfrak{h}_0^\vee$. Since $y^2 \in Z(G^\vee)$ (4) implies λ_0 is integral, so the inequality in (3) makes sense.

Alternatively if we fix an infinitesimal character χ we can define complete data to be triples (x, y, χ); this is defined to be (x, y, λ) where $\chi = \chi_\lambda$ and λ is dominant.

Now (x, y, λ_0) determines an $(\mathfrak{h}_0, \widetilde{H}_0^{\theta_x})$-module as follows. Take (λ_0, κ_0); we have to define κ_0. For this we need the basepoint.

Recall there is a surjection $p : X \to \mathcal{I}_W$ where $\mathcal{I}_W$ is the twisted involutions in the Weyl group. That is $W^\Gamma = W \rtimes \Gamma = \langle W, \delta \rangle$, and $\mathcal{I}_W = \{w\delta \,|\, (w\delta)^2 = 1\}$.

Suppose $w\delta \in \mathcal{I}_W$. Let $\Psi^+_{im,w}$ be the positive imaginary roots with respect to θ_{x,H_0} where $p(x) = w\delta$; this is independent of x.

Lemma 21 *There is a canonical way to choose, for all $w\delta \in \mathcal{I}_w$, an element $x[w]$ satisfying $p(x[w]) = w\delta$ and $\Psi^+_{im,w}$ is large with respect to $\theta_{x[w],H_0}$. In particular $x[1] = \exp(\pi i \rho^\vee \delta)$.*

Here is how $S = \{x[w] \,|\, w\delta \in \mathcal{I}_W\}$ is determined. Take $x[1] = \exp(\pi i \rho^\vee)\delta$. Consider the large block at ρ for the quasisplit form of G. Then S is the set of $x \in \mathcal{X}$ such that the corresponding standard representation for (the quasisplit form of) $G^\vee$ occurs in the character formula for the trivial representation.

Note that $x[w]^2 = z_\rho$ where $z_\rho = \exp(2\pi i \rho^\vee) \in Z(G)$.

Remark 22 The element $\exp(\pi i \rho^\vee)\delta$ is canonical and well defined, although the software does not tell us which number in the output of `kgb` it corresponds to. Making such an assignment amounts to choosing which parameter in the output of `block` corresponds to the trivial representation of $G^\vee$.

We apply this on the dual side. Given y, write $p(y) = w\delta^\vee$, and define $y[w]$ by the Lemma. Now write

$$y = \exp(2\pi i\gamma^\vee)y[w] \quad (\gamma^\vee \in X_*(H_0^\vee)\otimes\mathbb{C}). \tag{23}$$

Recall y defines the automorphism $\theta_{y,H_0^\vee}(h) = yhy^{-1}$ of $H_0^\vee$, and $y[w]$ defines the same automorphism of $H_0^\vee$. Denote this by $\theta^\vee$. Note that $y[w]^2 = \exp(2\pi i\rho)$.

Then define

$$\kappa_0 = \lambda_0 - (\gamma^\vee + \theta^\vee\gamma^\vee) \in \rho + X^*(H_0). \tag{24}$$

Let's check this. Note that $y[w]$

$$\begin{aligned} y^2 &= (\exp(2\pi i\gamma^\vee)y[w])^2 \\ &= \exp(2\pi i(\gamma^\vee + \theta^\vee\gamma^\vee))y[w]^2 \\ &= \exp(2\pi i(\gamma^\vee + \theta^\vee\gamma^\vee + \rho)) \end{aligned} \tag{25}$$

and setting this equal to $\exp(2\pi i\lambda_0)$ gives the assertion. Also (24) immediately implies

$$\lambda_0 - \theta^\vee\lambda_0 = \kappa_0 - \theta^\vee\kappa_0 \tag{26}$$

which is condition [5, Proposition 5.8(b)].

By the usual mumbo-jumbo (λ_0, κ_0) defines an $(\mathfrak{h}_0, \widetilde{H}_0^{\theta_x})$-module. See [3] and [5, Proposition 5.8]. Now this $(\mathfrak{h}, \widetilde{H}_0^{\theta_{x_0}})$-module defines a $(\mathfrak{g}, K_\xi)$-module where ξ is any element of $\widetilde{\mathcal{X}}$ lying over x. We eventually need to conjugate everything back to get a $(\mathfrak{g}, K_0)$-module, but we first state an elementary result.

Lemma 27 *Suppose (x, y, λ_0) is a complete parameter. Choose ξ lying over x, so this determines a $(\mathfrak{g}, K_\xi)$-module (π, V). Let (π^μ, V) be the $(\mathfrak{g}, K_{\mu(\xi)})$ module obtained in the usual way. Then the parameter for the representation (π^μ, V) is $(\mu(x), \mu^t(y), \mu^t(\lambda_0))$.*

This follows immediately by transport of structure.

Now assume x is conjugate to x_0. This is a well defined notion; it is equivalent to ξ is conjugate to ξ_0 where $\xi \in \widetilde{\mathcal{X}}$ lies over x. So choose $\xi \in \widetilde{\mathcal{X}}$ over x and $g \in G$ satisfying $g\xi_0 g^{-1} = \xi$. By Lemma 7 x corresponds to the K_0-orbit of $g^{-1}B_0 g$. So let:

$$
\begin{aligned}
B &= g^{-1}B_0g \\
H &= g^{-1}H_0g \\
\lambda &= g^{-1}\lambda_0 g \in \mathfrak{h}^* \\
\kappa &= g^{-1}\kappa_0 g \in \rho + X^*(H).
\end{aligned} \tag{28}
$$

Then (λ, κ) is a $(\mathfrak{h}, \widetilde{H}^{\theta_0})$-module, and this defines the $(\mathfrak{g}, K_0)$-module associated to the data (x, y, λ_0).

Proposition 29 *Fix ξ_0 and $K = K_{\xi_0}$ as above. Suppose (x, y, λ_0) is a complete parameter, and x is G-conjugate to x_0. Let (π, V) be the $(\mathfrak{g}, K_0)$-module defined by (x, y, λ_0).*

Assume $\mu(x_0) = x_0$. Then (π^μ, V) is given by the complete parameter $(\mu(x), \mu^t(y), \mu^t(\lambda_0))$.

Proof. This is merely a question of chasing around the definitions, using the preceding discussion to carry everything back to $(\mathfrak{g}, K_0)$. This is all pretty obvious with the possible exception of the κ term.

Let $\pi(x, y, \lambda_0)$ be the $(\mathfrak{g}, K_0)$ module defined by (x, y, λ_0). This is given by the data

$$
g^{-1}(\mathfrak{h}_0, \widetilde{H}_0^{\theta_{x_0}}, \lambda_0, \kappa_0)g. \tag{30)(a}
$$

Here $g^{-1}H_0g = H$, and see below for κ_0.

It is clear that $\pi(x, y, \lambda_0)^\mu$ is given by

$$
(\mu(g^{-1}\mathfrak{h}_0 g), \mu(g^{-1}(\widetilde{H}_0^{\theta_{x_0}})g), \mu^t(g^{-1}\lambda_0 g), \mu^t(g^{-1}\kappa_0 g)). \tag{30)(b}
$$

On the other hand the representation $\pi(\mu(x), \mu^t(y), \mu^t(\lambda_0))$ is given by

$$
u^{-1}(\mathfrak{h}_0, \widetilde{H}_0^{\theta_{x_0}}, \mu^t(\lambda_0), \kappa_0')u. \tag{30)(c}
$$

Here $u^{-1}H_0u = \mu(H)$. Using the fact that $\mu(H_0) = H_0$ and $g^{-1}H_0g = H$ we can take $u = \mu(g)$. Then (c) is equal to

$$
(\mu(g^{-1})\mathfrak{h}_0\mu(g), \mu(g^{-1})\widetilde{H}_0^{\theta_{x_0}}\mu(g), \mu(g^{-1})\mu^t(\lambda_0)\mu(g), \mu(g^{-1})\kappa_0'\mu(g)). \tag{30)(d}
$$

See below for κ_0'.

It is immediate that the first three terms are equal to the first three terms of (b):

$$\begin{aligned} \mu(g^{-1}\mathfrak{h}_0 g) &= \mu(g^{-1})\mathfrak{h}_0\mu(g) \\ \mu(g^{-1}(\widetilde{H}_0^{\theta_{x_0}})g) &= \mu(g^{-1})\widetilde{H}_0^{\theta_{x_0}}\mu(g) \\ \mu(g^{-1}\lambda_0 g) &= \mu(g^{-1})\mu(\lambda_0)\mu(g) \end{aligned} \tag{30)(e}$$

We have to show

$$\mu(g^{-1}\kappa_0 g) = \mu(g^{-1})\kappa_0'\mu(g), \tag{30)(f}$$

i.e.

$$\mu^t(\kappa_0) = \kappa_0'. \tag{30)(g}$$

Now we recall

$$\kappa_0 = \lambda_0 - (\gamma^\vee + \theta^\vee_{y,H_0^\vee}\gamma^\vee) \tag{30)(h}$$

with $y = \exp(2\pi i\gamma^\vee)y[w]$, where $p(y) = w\delta^\vee$.

On the other hand

$$\kappa_0' = \mu^t(\lambda_0) - (\beta^\vee + \theta^\vee_{\mu(y),H_0^\vee}\beta^\vee) \tag{30)(i}$$

with $\mu^t(y) = \exp(2\pi i\beta^\vee)y[w']$, where $p(\mu^t(y)) = w'\delta^\vee$. Clearly $w' = \mu^t(w)$.

This comes down to the following Lemma, restated for G.

Lemma 31 *Suppose $p(x) = w\delta$, so $p(\mu(x)) = \mu(w)\delta$. Then*

$$\mu(x[w]) = x[\mu(w)]. \tag{32}$$

This follows from Lemma 21, and the fact that $\mu(\exp(\pi i\rho^\vee)) = \exp(\pi i\rho^\vee)$ since μ preserves the positive roots. □

Remark 33 The Proposition actually shows the following. Suppose we're given a complete parameter (x, y, λ_0). Choose ξ lying over ξ, so this defines a $(\mathfrak{g}, K_\xi)$-module. Suppose $\mu(x)$ is G-conjugate to x. Then we can conjugate the $(\mathfrak{g}, K_{\mu(\xi)})$-module (π^μ, V) to a $(\mathfrak{g}, K_\xi)$ module. These two $(\mathfrak{g}, K_\xi)$-modules are isomorphic only if $\mu(x) = x$ (and also $\mu^t(y) = y, \mu^t(\lambda_0) = \lambda_0$).

Therefore it makes sense to assume that $\mu(x_0) = x_0$ for some x_0 which is G-conjugate to x.

7 The Chevalley Automorphism

For use in the next section here are some facts about the Chevalley automorphism.

Given G and a pinning, the Chevalley automorphism is the automorphism C satisfying $C(h) = h^{-1}$ for $h \in H$, and $C(X_\alpha) = X_{-\alpha}$ for all simple roots α. Recall the pinning includes the X_α for α simple, and $X_{-\alpha}$ is determined by $[X_\alpha, X_{-\alpha}] = \alpha^\vee$.

Lemma 34 *The Chevalley automorphism of G commutes with any distinguished automorphism of G.*

Now suppose we are given $G^\Gamma = \langle G, \delta \rangle$. Recall δ acts by a distinguished automorphism τ.

Definition 35 *Extend C to an automorphism of G^Γ by $C(\delta) = \delta$.*

It is clear that the automorphism C of G^Γ preserves $\widetilde{\mathcal{X}}$, and since C preserves H this action descends to $\mathcal{X}$.

It is well known that for g semisimple $C(g)$ is conjugate to g^{-1}. It is a remarkable fact that $C(x) = x^{-1}$ for all $x \in \mathcal{X}$.

Proposition 36 *We have $C(x) = x^{-1}$ for all $x \in \mathcal{X}$. More precisely for all $\xi \in \widetilde{\mathcal{X}}$ we have $C(\xi) = h\xi^{-1}h^{-1}$ for some $h \in H$.*

Proof.

First suppose $p(x) = \delta$. Choose $\xi = h\delta \in \widetilde{\mathcal{X}}$ lying over x. After conjugating by H we may assume $\tau(h) = h$, i.e. $h\delta = \delta h$. Then

$$C(\xi) = C(h\delta) = h^{-1}\delta = (\delta h)^{-1} = (h\delta)^{-1} = \xi^{-1}. \quad (37)$$

The condition $C(x) = x^{-1}$ is equivalent to $C(\xi) = h\xi^{-1}h^{-1}$, for any ξ lying over x, and for some h (depending on ξ). Writing $h\xi^{-1}h^{-1} = h\theta_x(h^{-1})\xi^{-1}$ we see the condition is equivalent to:

$$C(\xi)\xi \in A_x \quad (38)$$

where A_x is the identity component of $\{h \in H \,|\, \theta_x(h) = h^{-1}\}$.

We proceed by induction, using Cayley transforms in imaginary roots and Cayley transforms.

Recall the Tits group $\widetilde{W}$ is a subset of G, equipped with a map to the Weyl group W. It has generators σ_α for α simple, satisfying the braid relations, and $\sigma_\alpha^2 = m_\alpha = \alpha^\vee \in H$. Every element w of W has a canonical lift $\widetilde{w}$ to $\widetilde{W}$. It is immediate from the definition of the Tits group that if α is simple then

$$C(\sigma_\alpha) = \sigma_\alpha^{-1}. \tag{39}$$

Assume $C(\xi)\xi \in A_x$, so

$$C(\xi) = h\xi^{-1} \quad (h = t\theta_x(t^{-1}) \in A_x). \tag{40}$$

Suppose α is an x-imaginary root. Then $\sigma_\alpha\xi \in \mathcal{X}$, and $(\sigma_\alpha\xi)^2 = \xi^2$, which implies

$$\sigma_\alpha\xi\sigma_\alpha = \xi. \tag{41}$$

Then

$$\begin{aligned} C(\sigma_\alpha\xi)\sigma_\alpha\xi &= \sigma_\alpha^{-1}(h\xi^{-1})\sigma_\alpha\xi \\ &= \sigma_\alpha^{-1}h(\sigma_\alpha^{-1}\xi^{-1}\sigma_\alpha^{-1})\sigma_\alpha\xi \quad \text{(by (41))} \\ &= \sigma_\alpha^{-1}h\sigma_\alpha^{-1}. \end{aligned} \tag{42}$$

Now σ_α and h commute since $h \in A_x$ and α is x-imaginary. Therefore this equals $m_\alpha h$, which is contained in $A_{s_\alpha x}$. To be explicit, it equals

$$(ut)\theta_{s_\alpha x}(ut)^{-1} \tag{43}$$

where $u\theta_{s_\alpha x}(u^{-1}) = m_\alpha$, which is possible since $m_\alpha \in A_{s_\alpha x}$.

Now suppose α is any root and consider $\sigma_\alpha\xi\sigma_\alpha^{-1}$. The image x' of this element in $\mathcal{X}$ can be denoted $s_\alpha \times x$.

I claim

$$C(\sigma_\alpha\xi\sigma_\alpha^{-1})\sigma_\alpha\xi\sigma_\alpha^{-1} = u\theta_{s_\alpha\times x}(u^{-1}) \tag{44}$$

where $u = \sigma_\alpha t\sigma_\alpha^{-1}m_\alpha$.

You don't really want to check this, do you? Let's see, write $\sigma = \sigma_\alpha$ and $m = m_\alpha = \sigma_\alpha^2$:

$$\begin{aligned} C(\sigma\xi\sigma^{-1})\sigma\xi\sigma^{-1} &= \sigma^{-1}C(\xi)\sigma\sigma\xi\sigma^{-1} \\ &= \sigma^{-1}C(\xi)m\xi\sigma^{-1} \\ &= \sigma^{-1}C(\xi)\xi(\xi^{-1}m\xi)\sigma^{-1} \\ &= \sigma^{-1}[t\xi t^{-1}\xi^{-1}](\xi^{-1}m\xi)\sigma^{-1} \end{aligned} \tag{45}$$

On the other hand

$$u\theta_{s_\alpha\times x}(u^{-1}) = (\sigma t\sigma^{-1}m)(\sigma\xi\sigma^{-1})(\sigma t\sigma^{-1}m)^{-1}(\sigma\xi\sigma^{-1})^{-1}$$

(46)
$$\begin{aligned}
&= (\sigma t\sigma^{-1}m)(\sigma\xi\sigma^{-1})(m\sigma t^{-1}\sigma^{-1})(\sigma\xi^{-1}\sigma^{-1})\\
&= \sigma t(\sigma^{-1}m\sigma)\xi(\sigma^{-1}m\sigma)t^{-1}(\sigma^{-1}\sigma)\xi^{-1}\sigma^{-1}\\
&= \sigma tm\xi mt^{-1}\xi^{-1}\sigma^{-1}\\
&= \sigma t\xi(\xi^{-1}m\xi)mt^{-1}\xi^{-1}\sigma^{-1}\\
&= \sigma(t\xi t^{-1})(\xi^{-1}m)(\xi m\xi^{-1})\sigma^{-1}\\
&= \sigma[t\xi t^{-1}\xi^{-1}](m\xi m\xi^{-1})\sigma^{-1}\\
&= \sigma[t\xi t^{-1}\xi^{-1}](\xi m\xi^{-1})m\sigma^{-1}
\end{aligned}$$

Comparing these we need to show

(47) $$\sigma^{-1}[t\xi t^{-1}\xi^{-1}](\xi^{-1}m\xi)\sigma^{-1} = \sigma[t\xi t^{-1}\xi^{-1}](\xi^{-1}m\xi)m\sigma^{-1}.$$

Multiply on the left by σ^{-1} and the right by σ to give

(48) $$m[t\xi t^{-1}\xi^{-1}](\xi^{-1}m\xi) = \sigma[t\xi t^{-1}\xi^{-1}](\xi^{-1}m\xi)m$$

which is true since the terms in the middle are in H. □

We are also given $G^{\vee\Gamma} = \langle G^\vee, \delta^\vee\rangle$. Here $\delta^\vee$ acts by $\tau^\vee$. We recall the definition of $\tau^\vee$: the automorphism $-\tau^t$ of the (non-based) root datum of $G^\vee$ is defined. This induces an automorphism of the based root datum of $G^\vee$, and (via a chosen pinning) the automorphism $\tau^\vee$ of $G^\vee$.

The automorphism τ^t of the based root datum for $G^\vee$ also defines an automorphism denoted τ^t of $G^\vee$. Here is how $\tau^\vee$ and τ^t are related.

Lemma 49 *Let C be the Chevally automorphism of $G^\vee$ and let $\widetilde{w}_0$ be the lift of the long element of the Weyl group to $\widetilde{W}$ for $G^\vee$. Then*

(50) $$\tau^\vee(g) = \widetilde{w}_0 C(\tau^t(g))\widetilde{w}_0^{-1}.$$

It is worth mentioning that $\widetilde{w}_0^2 = \exp(2\pi i\rho)$ (a proof of this was supplied by John Stembridge and (independently) Marc van Leeuwen).

8 Case of the Hermitian Dual

Fix $G^\Gamma = \langle G, \delta\rangle$, where δ acts by τ, and fix K in this inner class.

Proposition 51 *Suppose (π, V) is an irreducible $(\mathfrak{g}, K)$-module with real infinitesimal character. Then (π^h, V^h) is isomorphic to (π^τ, V).*

Sketch of Proof. Let θ_π be the global character of π. It is well known that if π^* is the contragredient then

(52)(a) $$\theta^*_\pi(g) = \theta_\pi(g^{-1}).$$

There is also a bar-operation on representations satisfying

(52)(b) $$\theta_{\overline{\pi}}(g) = \overline{\theta_\pi(g)}.$$

There is a basic result that $\pi^h \simeq \overline{\pi}^*$ [reference?] so

(52)(c) $$\theta_{\pi^h}(g) = \overline{\theta_\pi(g^{-1})}.$$

Let's assume that G is acceptable, i.e. ρ factors to H. Let λ be the infinitesimal character of π. By results of Harish-Chandra and the Casselman-Osborn conjecture, for any real Cartan subgroup $H(\mathbb{R})$ we can write

(52)(d) $$\theta_\pi(g) = \frac{\sum_\chi a_\chi \chi(g)}{D(g)}.$$

Here g is contained in a certain subset of the regular elements of $H(\mathbb{R})$. The sum is over characters χ satisfying $d\chi$ is conjugate to λ, and the a_χ are integers. Finally $D(g)$ is a version of the Weyl denominator. See [1, Section 4].

If λ is real then $d\chi$ is real for all χ occuring in this formula. That is, if we write the real Lie algebra of $H(\mathbb{R})$ as $\mathfrak{t}_0 + \mathfrak{a}_0$, then $d\chi \in i\mathfrak{t}_0 + \mathfrak{a}_0$ for all χ occuring in the sum. Therefore for any such χ, writing $g = ta$, $\chi(ta) = \chi(t)\chi(a)$ with $|\chi(t)| = 1$ and $\chi(a) \in \mathbb{R}$. Then

(52)(e) $$\begin{aligned} \overline{\chi(g^{-1})} &= \overline{\chi(t^{-1})\chi(a^{-1})} \\ &= \overline{\chi(t)}^{-1}\overline{\chi(a)^{-1}} \\ &= \chi(t)\chi(a^{-1}) \end{aligned}$$

A similar result holds for $D(g)$: $\overline{D(g^{-1})} = D(ta^{-1})$.

Now we claim that for $g = ta \in H(\mathbb{R})$, $\tau(g)$ is $G(\mathbb{R})$-conjugate to ta^{-1}. This requires some thought, since it involves $G(\mathbb{R})$, not $G(\mathbb{C})$, but let's assume it for now. Then

$$\theta_{\pi^h}(g) = \overline{\theta_\pi(g^{-1})} = \frac{\sum_\chi \chi(ta^{-1})}{D(ta^{-1})} \tag{52(f)}$$

and

$$\theta_{\pi^\tau}(g) = \theta_\pi(\tau(g))\frac{\sum_\chi \chi(ta^{-1})}{D(ta^{-1})} \tag{52(g)}$$

□

Remark 53 If G is not acceptable there is the usual issue with ρ-covers, but nothing essentially different.

By Proposition 29 we conclude:

Corollary 54 *Define $G^{\Gamma,\dagger}$ by taking $\mu = \tau$. Fix $x_0 \in \mathcal{X}$, an inverse image $\xi_0 \in \widetilde{\mathcal{X}}$, and corresponding $K = K_{\xi_0}$. Assume $\tau(x_0) = x_0$. Suppose (x, y, λ) is a complete parameter, with corresponding $(\mathfrak{g}, K)$-module (π, V). Assume λ is real. Then (π^h, V) is given by parameter $(\tau(x), \tau^t(y), \tau^t(\lambda))$.*

Remark 55 Note that this doesn't involve any any computation in $\mathcal{Z}$. We give a direct atlas-theoretic proof below.

Remark 56 The assumptions that $\tau(\xi_0) = \xi_0$ and x is G-conjugate to $x_0 = p(\xi_0)$ implies that $\tau(x)$ is G-conjugate to x_0. Therefore (in spite of superficial appearances) $(\tau(x), \tau^t(y), \tau^t(\lambda))$ also necessarily defines a $(\mathfrak{g}, K)$-module.

Here is a sketch of a purely atlas-theoretic proof of Corollary 54. In this section only we denote our fixed Cartan and Borel by H, B (rather than H_0, B_0 elsewhere).

Suppose (x, y, λ) is a complete parameter for the $(\mathfrak{g}, K)$-module (π, V). We assume λ is real, i.e. $\lambda \in X^*(H) \otimes \mathbb{R}$. Let $\overline{\lambda}$ be the complex conjuate of λ with respect to the real form of H defined by θ_x. This differs from the one defined by $X^*(H) \otimes \mathbb{R}$ by -1 on the $\mathfrak{t}$-part. See [2]. An important fact is that for λ real,

$$-\overline{\lambda} = \theta^t(\lambda). \tag{57}$$

(the ^{t} denotes transpose, and is only there because $\lambda \in \mathfrak{h}^*$, not $\mathfrak{h}$).

Lemma 58 *Write $p(x) = w_x\delta \in \mathcal{I}_W$, so $w_x \in W$. Let $w_x^\vee$ be the corresponding element on the dual side. The parameter for (π^h, V^h) is*

$$(w_x^{-1}xw_x, w_x^{\vee -1}y^{-1}w_x^\vee, -w_x^{\vee -1}\overline{\lambda}). \tag{59}$$

Here $w_x^{-1}xw_x$ is the cross action of W on $\mathcal{X}$; more precisely this is $n_x^{-1}xn_x$ where $n_x \in G$ is a representative of $w_x \in W$.

Proof. It is straightforward that the map of the Weil group is given by the parameter $(x, y^{-1}, -\overline{\lambda})$. See [2]. This isn't a complete parameter if $-\overline{\lambda}$ is not dominant. Write $x = g_x\delta$, and $p(x) = w_x\delta \in \mathcal{I}_W$, so $p(g_x) = w_x$. Then

$$\theta_x(X) = g_x\tau(X)g_x^{-1} = w_x(\tau(X)) \quad (X \in \mathfrak{h}_0). \tag{60)(a}$$

and acting on $\mathfrak{h}^*$ we have

$$\theta_x^t(\lambda) = w_x^\vee(\tau^t(\lambda)) \quad (\lambda \in \mathfrak{h}_0^*). \tag{60)(b}$$

Therefore by (57)

$$-\overline{\lambda} = \theta_x^t(\lambda) = w_x^\vee(\tau^t(\lambda)) \tag{60)(c}$$

and multiplying both sides on the left by $w_x^{\vee -1}$ gives

$$-w_x^{\vee -1}\overline{\lambda} = \tau^t(\lambda). \tag{60)(d}$$

Now since τ is distinguished $\tau^t(\lambda)$ is dominant.

Conjugating x by w_x^{-1} and $y, -\overline{\lambda}$ by $w_x^{\vee -1}$ gives (59). □

Lemma 61 *We can also write* (59) *as*

$$(\tau(x), w_x^{\vee -1}y^{-1}w_x^\vee, \tau^t(\lambda)). \tag{62}$$

Proof. The last entry is given by (60). For the first, recall g_x maps to w_x, so

$$w_x^{-1}xw_x = g_x^{-1}(g_x\delta)g_x = \delta g_x = \tau(g_x)\delta = \tau(x). \tag{63}$$

□

The hard part is:

Lemma 64

(65) $$w_x^{\vee -1}\, y^{-1}\, w_x^{\vee} = \tau^t(y).$$

Proof.

First of all we claim if $p(y) = w_y^{\vee}$ then

(66) $$w_x^{\vee} = w_y^{\vee} w_0^{\vee}$$

where $w_0^{\vee}$ is the long element of the Weyl group of $G^{\vee}$. To see this note that for $X \in \mathfrak{h}$ $\theta_x(X) = w_x \tau(X)$, so for $X^{\vee}$ in $\mathfrak{h}^{\vee}$, $\theta_x^t(X^{\vee}) = \theta_x^{\vee}\tau^t(X^{\vee})$. This is required to equal $\theta_y(X^{\vee}) = w_y^{\vee}\tau^{\vee}(X^{\vee})$, i.e.

(67) $$w_x^{\vee}\tau^t(X^{\vee}) = -w_y^{\vee}\tau^{\vee}(X^{\vee}).$$

By Lemma 49 $\tau^{\vee}(X^{\vee}) = -w_0^{\vee}\tau^t(X^{\vee})$, since the Chevalley automorphism acts by -1 on $\mathfrak{h}^{\vee}$. This gives the claim.

Now choose an inverse image $\xi^{\vee} = g_y^{\vee}\tau^{\vee}$ of y in $\widetilde{\mathcal{X}}$, and let $\sigma_0^{\vee} \in G^{\vee}$ be a representative of the long element of the Weyl group. Note that $g_y^{\vee}$ is a representative of $w_y^{\vee}$. Then by (66) $g_y^{\vee}\sigma_0^{\vee}$ is a representative of $w_x^{\vee}$, so $w_x^{\vee -1} y w_x^{\vee}$ is the image of

(68) $$\sigma_0^{\vee -1} g_y^{\vee -1} \xi_y^{-1} g_y^{\vee} \sigma_0^{\vee}.$$

Writing $\xi^{\vee} = g_y^{\vee}\delta^{\vee}$ gives

(69) $$\sigma_0^{\vee -1} g_y^{\vee -1}\delta^{\vee} g_y^{\vee -1} g_y^{\vee}\sigma_0^{\vee}. = \sigma_0^{\vee -1} g_y^{\vee -1}\delta^{\vee}\sigma_0^{\vee}$$

Note that $\tau^{\vee}(\xi^{-1}) = \tau^{\vee}(\delta^{\vee} g_y^{\vee -1}) = g_y^{\vee -1}\delta^{\vee}$, so this equals

(70) $$\sigma_0^{\vee -1}\tau^{\vee}(\xi_y^{-1})\sigma_0^{\vee}$$

By Lemma 49 this equals

(71) $$\sigma_0^{\vee -1}(\sigma_0^{\vee} C(\tau^t(\xi_y^{-1})\sigma_0^{\vee -1})\sigma_0^{\vee} = \tau^t(C(\xi_y^{-1})).$$

By Proposition 36 $C(\xi_y^{-1}) = \xi_y$ modulo conjugation by $H^{\vee}$, i.e. the image of $C(\xi_y^{-1})$ in $\mathcal{X}^{\vee}$ is y. □

Proposition 72 *Suppose (π, V) has parameter (x, y, λ) and λ is real. Then (π^h, V) has parameter*

$$(\tau(x), \tau^t(y), \tau^t(\lambda)). \tag{73}$$

This agrees with Corollary 54.

Remark 74 Strictly speaking parameters are only defined if λ is integral, which (ignoring split tori in the center) implies λ is real. Presumably once we've extended the atlas construction to general infinitesimal character this result will hold.

9 Automorphism of the Weil group

This section isn't really needed, but it has some philosophical appeal.

There are two interesting automorphisms of the Weil group. It is worthwhile to compute the effect of these automorphisms on L-packets.

Recall $W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle$ where $jzj^{-1} = \overline{z}$ and $j^2 = -1$.

Definition 75 *Define automorphism α, β of $W_{\mathbb{R}}$ as follows. Let $\alpha(j) = \beta(j) = j$. For $z \in \mathbb{C}^*$ define $\alpha(z) = z^{-1}$ and $\beta(z) = \overline{z}^{-1}$.*

For $\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}$ let Π_ϕ denote the corresponding L-packet of a real form of G.

Lemma 76 *Suppose $\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}$ is an L-homomorphism. Let ϕ_α be the L-homomorphism $\phi_\alpha(w) = \phi(\alpha(w))$, and define ϕ_β similarly.*
(1) $\Pi_{\phi_\alpha} = \Pi_\phi^* = \{\pi^* \mid \pi \in \Pi_\phi\}$.

(2) Assume Π_ϕ has real infinitesimal character. Then

$$\Pi_{\phi_\beta} = \Pi_\phi^h = \{\pi^h \mid \pi \in \Pi_\phi\}. \tag{77}$$

Sketch of Proof. Suppose ϕ is given by parameter (y, λ). Then

(78)(a)
$$\begin{aligned} \phi_\alpha &\leftrightarrow (y^{-1}, -\lambda) \\ \phi_\beta &\leftrightarrow (y^{-1}, -\theta_y(\lambda)) \end{aligned}$$

It is straightforward to see that ϕ_α corresponds to the contragredient. For the second part we need (57), which says that for λ real, $-\overline{\lambda} = \theta^t(\lambda)$, so

(78)(b) $$\theta_y(\lambda) = \overline{\lambda}.$$

More details later...

□

10 Stabilizer in the Extended Group

As in Section 6 fix ξ_0 and let $\theta_0 = \theta_{\xi_0}$, $K_0 = K_{\xi_0}$. Let $x_0 = p(\xi_0)$ and we assume $\mu(x_0) = x_0$ (see Remark 33). As in Section 1 we're interested in $(\mathfrak{g}, K_0^\dagger)$-modules where $K_0^\dagger = \mathrm{Cent}_{G^\dagger}(\xi_0)$. So for $\xi \in \widetilde{\mathcal{X}}$ let

(79) $$K_\xi^\dagger = \mathrm{Cent}_{G^\dagger}(\xi).$$

We start with an element $x \in \mathcal{X}$ which is G-conjugate to $x_0 = p(\xi_0)$, which determines a K_0-orbit on G/B. Recall (Lemma 7) if ξ lies over x, and $\xi = g\xi_0 g^{-1}$, then this is the K_0-orbit of $B = g^{-1}B_0 g$. We need to compute $\mathrm{Stab}_{K_0^\dagger}(B)$.

If $\mu(x) \neq x$ then $\mathrm{Stab}_{K_0^\dagger}(B) = \mathrm{Stab}_{K_0}(B)$, so assume $\mu(x) = x$. By Proposition 29 this is necessary for this orbit to support a representation fixed by μ.

Since $\mu(x_0) = x_0$ we know $\mu(\xi_0) = t\xi_0 t^{-1}$ for some $t \in H_0$. So here is the situation:

(80)(a) $$\mu(\xi_0) = t\xi_0 t^{-1} \quad (t \in H_0)$$
(80)(b) $$\mu(x) = x$$
(80)(c) $$\xi = g\xi_0 g^{-1}$$
(80)(d) $$B = g^{-1}B_0 g$$

By (b) we have

(80)(e) $$\mu(\xi) = h^{-1}\xi h \quad (\text{some } h \in H_0).$$

By (c) and (e) $\mu(g\xi_0 g^{-1}) = h^{-1}(g\xi_0 g^{-1})h$, and by (a) this implies

(80)(f) $$g^{-1}h\mu(g)t \in K_0.$$

Now

$$\begin{aligned}\mathrm{Stab}_{K_0^\dagger}(B) &= \mathrm{Stab}_{K_0^\dagger}(g^{-1}B_0g)\\ &= g^{-1}(\mathrm{Stab}_{gK_0^\dagger g^{-1}}(B_0))g.\end{aligned} \tag{81}$$

Recall $K_0^\dagger = \mathrm{Stab}_{G^\dagger}(\xi_0)$, so $gK_0^\dagger g^{-1} = \mathrm{Stab}_{G^\dagger}(\xi) = K_\xi^\dagger$ (cf. (79)). Therefore

$$\mathrm{Stab}_{K_0^\dagger}(B) = g^{-1}(\mathrm{Stab}_{K_\xi^\dagger}(B_0))g. \tag{82}$$

Lemma 83 *With $h \in H_0$ as in* (80)(e) *we have:*

$$\mathit{Stab}_{K_\xi^\dagger}(B_0) = \langle H_0^{\theta_x}, h\epsilon\rangle. \tag{84}$$

Remark 85 Note that the second term is contained in $H_0\epsilon$.

Proof. The first term is the usual. For the second,

$$\begin{aligned}(h\epsilon)B_0(h\epsilon)^{-1} &= h\mu(B_0)h^{-1}\\ &= hB_0h^{-1} \quad \text{(since } \mu \text{ is distinguished)}\\ &= B_0 \quad \text{(since } h \in H_0 \subset B_0).\end{aligned} \tag{86}$$

Also

$$\begin{aligned}(h\epsilon)\xi(h\epsilon)^{-1} &= h\mu(\xi)h^{-1}\\ &= h(h^{-1}\xi h)h^{-1} \quad \text{(by (80)(e))}\\ &= \xi\end{aligned} \tag{87}$$

so $h\epsilon \in K^\dagger$. □

Lemma 88

$$\mathit{Stab}_{K_0^\dagger}(B) = \langle H^{\theta_{x_0}}, g^{-1}h\mu(g)\epsilon\rangle. \tag{89}$$

This is immediate from the previous Lemma, since $g^{-1}(h\epsilon)g = g^{-1}h\mu(g)\epsilon$. Note that $g^{-1}h\mu(g)\epsilon \in K_0^\dagger$ (this is obvious, but's let's double-check just to make sure):

$$\begin{aligned}(g^{-1}h\mu(g)\epsilon)\xi_0(g^{-1}h\mu(g)\epsilon)^{-1} &= g^{-1}h\mu(g)\mu(\xi_0)(g^{-1}h\mu(g))^{-1}\\ &= g^{-1}h\mu(g)t\xi_0t^{-1}(g^{-1}h\mu(g))^{-1}\\ &= (g^{-1}h\mu(g)t)\xi_0(g^{-1}h\mu(g)t)^{-1} = \xi_0\end{aligned} \tag{90}$$

where the last equality is by (80)(f). Also note that $g^{-1}h\mu(g)\epsilon = (g^{-1}hg)(g^{-1}\epsilon g) \in H(g^{-1}\epsilon g)$, but this is not necessarily in $H\epsilon$.

11 Parameters for $(\mathfrak{g}, K^\dagger)$-modules

As in Section 6 fix ξ_0 and let $\theta_0 = \theta_{\xi_0}$, $K_0 = K_{\xi_0}$. Assume $\mu(\xi_0) = \xi_0$.

As in the previous section fix $x \in \mathcal{X}$ with $\mu(x) = x$. Choose ξ lying over x. As in Lemma 83 we have

$$\mathrm{Stab}_{K_\xi^\dagger}(B_0) = \langle H_0^{\theta_x}, h\epsilon\rangle \tag{91}$$

where $h \in H_0$ satisfies $\mu(\xi) = h^{-1}\xi h$ (cf. (80)(e)).

Suppose (λ_0, κ_0) is an $(\mathfrak{h}_0, \widetilde{H}_0^{\theta_x})$-module. Recall (Section 6) $\kappa_0 \in \rho + X^*(H_0)$. Note that μ acts on $(\mathfrak{h}_0, \widetilde{H}_0^{\theta_x})$ (for the cover this uses that μ is distinguished) and $(\lambda_0, \kappa_0)^\mu = (\mu^\dagger(\lambda_0), \mu^\dagger(\kappa_0))$.

This is as far as I've gotten for now…

Never mind the cover, an $(\mathfrak{h}, \langle H_0^{\theta_x}, h\epsilon\rangle)$-module (which restricts irreducibly) is an $(\mathfrak{h}, H_0^{\theta_x})$-module (λ_0, κ_0), together with a complex number z satisfying

$$z^2 = \kappa_0(h\mu(h)). \tag{92}$$

We need to figure out nice parameters for this. We then need to incorporate the ρ-cover of H. Finally (recall $\mathrm{Stab}_{K_0^\dagger}(B) = \langle H^{\theta_{x_0}}, g^{-1}h\mu(g)\epsilon\rangle$) we'll conjugate to get an $(\mathfrak{h}, \langle H^{\theta_{x_0}}, g^{-1}h\mu(g)\epsilon\rangle)$-module (some cover of this).

References

[1] J. Adams. Computing character using atlas. preprint, www.liegroups.org/papers.

[2] J. Adams. Notes on the hermitian dual. preprint, www.liegroups.org/papers.

[3] J. Adams. Some notes on parametrizing representations. preprint, www.liegroups.org/papers.

[4] Jeffrey Adams and F. du Cloux. Algorithms for representation theory of real reductive groups. preprint, to appear in J. Inst. Math. Jussieu, www.liegroups.org/papers.

[5] D. Vogan J. Adams. L-groups, projective representations, and the Langlands classification. 113:45–138, 1992.