Discrete Series and Characters of the Component Group

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Suppose $G(\mathbb{R})$ is a real reductive group, with L-group ${}^\vee G^\Gamma$, and $\phi : W_\mathbb{R} \to {}^\vee G^\Gamma$ is an L-homomorphism. There is a close relationship between the L-packet associated to ϕ and characters the component group of the centralizer of the image of ϕ. This was originally observed by Shelstad [10], and further refined by Arthur, Kottwitz, Langlands and Shelstad. This is reinterpreted in [3], to (among other things) make it a canonical bijection. This involves a number of changes, including using the notion of strong real form, several strong real forms at once, and taking a cover of the component group. This cover is not necessarily a two-group, so the values of the character may not be just signs.

An important special case is that of discrete series L-packets. It is worthwhile to make the correspondence between characters of (covers of) the centralizer and the (generalized) L-packet in this case. While this is a special case of [3], it isn't so easy to extract it, or even from the more elementary reference [1].

These notes consist of a mostly self-contained treatment of this special case, with some details in the case of unitary groups. We also address a question raised by Michael Harris about endoscopy.

While the primary reference is *The Langlands Classification and Irreducible Characters* [3], for what is in this paper other more accessible references are sufficient. *Lifting of Characters* [1], which concentrates on the case of regular integral infinitesimal character has most of what is needed. Some of the approach here follows that of *Algorithms for Representation of Real Reductive Groups* [4] , for which *Guide to the Atlas Software* [2] is a good introduction.

1 Basic Setup

Our starting point is a connected, complex, reductive group G. We begin with some definitions independent of any real structure.

We fix Cartan and Borel subgroups $H \subset B$ of G. Let $G^\vee$ be the (connected, complex) dual group, and fix Cartan and Borel subgroups $H^\vee \subset B^\vee$ of $G^\vee$. By construction we have canonical identifications

$$X^*(H) = X_*(H^\vee), \quad X_*(H) = X^*(H^\vee) \tag{1.1}$$

where X^* and X_* denote the character and co-character lattices, respectively. Let R and $R^\vee$ be the root and coroot lattices, respectively. We write $R(G,H), R^\vee(G,H)$ etc. if it is necessary to specify the extra data. Define the weight and coweight lattices $P, P^\vee$:

$$\begin{aligned} P &= \{\lambda \in X^*(H) \otimes \mathbb{R} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \quad \text{for all } \alpha^\vee \in R^\vee\} \\ P^\vee &= \{\gamma^\vee \in X_*(H) \otimes \mathbb{R} \mid \langle \alpha, \gamma^\vee \rangle \in \mathbb{Z} \quad \text{for all } \alpha \in R\}. \end{aligned} \tag{1.2}$$

These are lattices if G is semisimple, and in general contain real vector spaces of dimension $\dim(Z)$, where $Z = Z(G)$ is the center of G. For the choice of $\mathbb{R}$ here (versus $\mathbb{Q}$ or $\mathbb{C}$) see Definition 2.1. Let W be the Weyl group. We have:

$$\begin{aligned} P^\vee &= \{\gamma^\vee \in X_*(H) \otimes \mathbb{R} \mid \exp(2\pi i \gamma^\vee) \in Z\} \\ X_*(H) &= \{\gamma^\vee \in X_*(H) \otimes \mathbb{R} \mid \exp(2\pi i \gamma^\vee) = 1\}. \end{aligned} \tag{1.3}$$

We're interested in representations of real forms of G. A *real form* of G is the fixed points $G(\mathbb{R})$ of an anti-holomorphic involution σ of G. There is a *holomorphic* involution θ of G commuting with σ, such that $G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$. This is the *Cartan involution* of G corresponding to $G(\mathbb{R})$. The complexification of $G(\mathbb{R})^\theta$ is G^θ, and is denoted K. This is a complex reductive group (possibly disconnected).

For example if $G = \mathrm{GL}(n, \mathbb{C})$ and $\sigma(g) = \overline{g}$ (complex conjugate of coordinates), then $G(\mathbb{R}) = \mathrm{GL}(n, \mathbb{R})$. The maximal compact subgroup of $\mathrm{GL}(n, \mathbb{R})$ is $K(\mathbb{R}) = O(n)$. This is the fixed points of $\theta(g) = {}^t g^{-1}$, and $K = G^\theta = O(n, \mathbb{C})$.

The classification of real forms of G can be stated entirely in terms of the (holomorphic) Cartan involutions θ. For $g \in G$ write $\mathrm{int}(g)$ for the inner automorphism $\mathrm{int}(g)(h) = ghg^{-1}$.

Definition 1.4 *A real form of G is a holomorphic involution θ of G. Two real forms θ_1, θ_2 are said to be equivalent if $\theta_2 = int(g)\theta_1 int(g^{-1})$ for some $g \in G$.*

For the relation with the standard definition of real forms see [4, Section 3] and the references there.

For example $\theta = 1$ is the *compact* real form of G: $K = G$, so $G(\mathbb{R}) = K(\mathbb{R})$ is compact.

It is natural to consider multiple real forms simultaneously; not all real forms but those in a given *inner class*. Write $\mathrm{Aut}(G), \mathrm{Int}(G)$ for the groups of (holomorphic) automorphisms of G and inner automorphisms, respectively.

Definition 1.5 *Two real forms θ_1, θ_2 are in the same inner class if θ_1, θ_2 have the same image in $Out(G) = Aut(G)/Int(G)$, i.e. $\theta_2 = int(g) \circ \theta_1$ for some $g \in G$.*

Fix an involution $\gamma \in Out(G)$. The associated inner class of real forms of G is the set of involutions $\theta \in Aut(G)$ whose image in $Out(G)$ is γ.

This is equivalent to the usual notion of inner class. For more details see [4, Section 3].

The collection of real forms which have discrete series representations forms a single inner class. The next result is standard; the only hard part is (4), which is a basic result of Harish-Chandra.

Lemma 1.6 *Suppose $G(\mathbb{R})$ is a real form of G. Let θ be a corresponding Cartan involution of G, and let $K = G^\theta$. The following conditions are equivalent:*

1. *$G(\mathbb{R})$ contains a compact Cartan subgroup;*

2. *$rank(G) = rank(K)$;*

3. *θ is an inner involution;*

4. *$G(\mathbb{R})$ has discrete series representations.*

This is the *equal rank* inner class of real forms of G. It is also known as the *compact* inner class, since it is the inner class of the compact real form $\theta = 1$.

Associated to an inner class of real forms of G is the *extended group* $G^\Gamma = G \rtimes \Gamma$, where $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. For example see [4, Section 5]. The action of Γ is by the Cartan involution of the most compact real form in the inner class. In the case of the compact inner class this action is trivial, so $G^\Gamma = G \times \Gamma$ and we may safely drop Γ from the notation. Many constructions of [1], [3] and [4] involving the "twist" simplify in this setting.

2 Strong real forms

We turn next to a refinement of the notion of real form which plays an important role. We emphasize that we work entirely in the equal rank inner class, i.e. the Cartan involutions are inner (cf. Lemma 1.6).

There is one technical point which arises only if G is not semisimple. An element g of G is said to be *elliptic* if the closure in the analytic topology of the cyclic subgroup generated by g is compact [3, Definition 22.1]. If g has finite order then it is elliptic.

Definition 2.1 *A strong involution of G (in the equal rank inner class) is an elliptic element $x \in G$ satisfying $x^2 \in Z$. A strong real form of G (in the equal rank inner class) is a conjugacy class of strong involutions. If x is a strong involution let $\theta_x = int(x)$ and $K_x = G^{\theta_x} = Cent_G(x)$.*

The map $x \to \theta_x$ is a surjection

(2.2)(a) $$\{\text{strong involutions}\} \twoheadrightarrow \{\text{involutions}\}$$

and factors to a surjection

(2.2)(b) $$\{\text{strong real forms of } G\} \twoheadrightarrow \{\text{real forms of } G\}$$

(the involutions in (a) are inner, and the real forms in (b) are equal rank). If G is adjoint (b) is a bijection.

In [3] and elsewhere we make the stronger assumption that x^2 has finite order. The elliptic condition makes the construction of Section 6 easier. If G is semisimple the elliptic condition is empty since Z is finite.

To emphasize the distinction with strong real forms, we sometimes refer to the real forms of Definition 1.4 as *weak real forms.*

From (1.3) we have

Lemma 2.3 *Let $\mathcal{X}_1 = \{x \in H \,|\, x^2 \in Z, x$ is elliptic$\}$. There are canonical bijections*

$$\text{(2.4)(a)} \qquad \mathcal{X}_1 \overset{1-1}{\longleftrightarrow} P^\vee/2X_*(H)$$

and

$$\text{(2.4)(b)} \qquad \{\textit{strong real forms of } G\} \overset{1-1}{\longleftrightarrow} \mathcal{X}_1/W.$$

The set $\mathcal{X}_1$ is a special case of the set $\mathcal{X}_\tau$ (with $\tau = 1$) of [4, Section 11].
Proof. First assume G is semisimple. The map

$$\text{(2.5)} \qquad P^\vee \ni \gamma^\vee \to \exp(\pi i \gamma^\vee) \in \mathcal{X}_1$$

factors to the bijection (a). Every strong involution is conjugate to an element of H, i.e. an element of $\mathcal{X}_1$; two such elements are conjugate if and only if they are conjugate by W. Together with (a) this gives (b).

The proof in general is the same, using the fact [3, Lemma 22.2] that $x \in H$ is elliptic if and only if $x = \exp(X)$ for $X \in X_*(H) \otimes \mathbb{R}$ (cf. 1.2). ∎

It is often convenient consider only those x with x^2 fixed; this is always a finite set. So fix $z \in Z$ and let

$$\text{(2.6)} \qquad \mathcal{X}_1[z] = \{x \in H \,|\, x^2 = z\}.$$

We refer to $\mathcal{X}_1[z]$ (resp. $\mathcal{X}_1[z]/W$) as the as the strong involutions (resp. strong real forms) of type z.

For $w \in Z$ the map $x \to wx$ is a bijection between $\mathcal{X}_1[z]$ and $\mathcal{X}_1[w^2 z]$. In particular these two spaces map to the same set of weak real forms via (2.2). For this reason it is useful to cut down the space $\mathcal{X}_1$ to the *reduced* parameter space $\mathcal{X}_1^r$: choose a set of representative $\{z_1, \dots, z_n\}$ of Z/Z^2, and define

$$\text{(2.7)} \qquad \mathcal{X}_1^r = \bigcup_i \mathcal{X}_1[z_i].$$

The map from X_1^r to weak real forms is surjective. See [4, Section 13].

An important special class of strong real forms are the following. Let $z(\rho^\vee) = \exp(2\pi i \rho^\vee)$ where $\rho^\vee = \frac{1}{2}\sum \alpha^\vee \in P(G^\vee, H^\vee)$; the sum is over any set of positive roots, and is independent of this choice. Following [15, Definition 2.6] we say a strong involution or real form is *pure* if $x \in \mathcal{X}_1[z(\rho^\vee)]$.

Example 2.8 Suppose $x \in Z$. Then $\theta_x = 1$ and $K = G$, so this maps the compact weak real form of G. Thus the fiber of the map from strong real forms to the compact weak real form is Z.

At the other extreme let $x = \exp(\pi i \rho^\vee)$. We will see later (5.9) that θ_x is the Cartan involution of a quasisplit real form.

3 Representations of strong real forms

Suppose for the moment that $G(\mathbb{R})$ is a real form of G, with Cartan involution θ and complexified maximal compact subgroup $K = G^\theta$. Let $\mathfrak{g}$ be the (complex) Lie algebra of G. A $(\mathfrak{g}, K)$-modules is a vector space V, equipped with compatible actions of $\mathfrak{g}$ and K. For more details, and the relation with representations of $G(\mathbb{R})$, see [14].

Return now to the setting of the previous section. A *representation of a strong involution* is a pair (x, π) where x is a strong involution and π is a $(\mathfrak{g}, K_x)$-module.

Some care is required to to allow a notion of equivalence which allows conjugation by G. We say (x, π) is equivalent to (x', π') if there exists $g \in G$ such that $gxg^{-1} = x'$ and $\pi' \simeq \pi^g$. Write $[x, \pi]$ for the equivalence class of (x, π). A *representation of a strong real form* of G is an equivalence class of representations of strong involutions. Here is a key example.

Example 3.1 Let $G = \mathrm{SL}(2, \mathbb{C})$ and $x = \mathrm{diag}(i, -i)$. Then $K_x = H = \mathbb{C}^\times$ and the corresponding real group is $SU(1,1) \simeq \mathrm{SL}(2, \mathbb{R})$. There is a unique irreducible $(\mathfrak{g}, K_x)$-module whose restriction to K_x is the characters $\{z^k \mid k = 2, 4, 6, \dots\}$. This is the $(\mathfrak{g}, K_x)$-module of a discrete series representation of $\mathrm{SL}(2, \mathbb{R})$.

Suppose g is a representative of the nontrivial element of the Weyl group. Then $\mathrm{int}(g)$ is an automorphism of $(\mathfrak{g}, K_x)$, which acts by $h \to h^{-1}$ on K_x, and therefore takes π to the contragredient $\overline{\pi}$. However we do *not* want to consider π and $\overline{\pi}$ to be equivalent.

This is taken care of by the fact that we keep track not only of K_x but of the strong involution x. Replace π with $[x, \pi]$, the equivalence class of (x, π). Then $\mathrm{int}(g)(x) = -x$, so $[x, \pi] = [int(g)x, \pi^g] = [-x, \overline{\pi}]$, but $[x, \pi] \neq [x, \overline{\pi}]$.

Note that $\{\pi, \overline{\pi}\}$ is an L-packet of discrete series representations. Using strong real forms this is $\{[x, \pi], [x, \overline{\pi}]\}$. We prefer to think of this as

$$\{[x, \pi], [-x, \pi]\}, \tag{3.2}$$

which is parametrized by the set $\{x, -x\}$ of strong involutions. This is a special case of the general situation; see Proposition 5.3.

4 L-parameters

We continue to work with the equal rank inner class of real forms of G. The L-group of a real form only depends on its inner class; let ${}^\vee G^\Gamma$ be the L-group of the equal rank inner class. Basic references for this section are [5] and [8]; also see [4, Section 7].

Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$, and suppose ϕ is an admissible homomorphism of $W_{\mathbb{R}}$ into ${}^\vee G^\Gamma$. Fix a strong involution x. Langlands associates to x and ϕ a (possibly empty) L-packet denoted $\Pi(x, \phi)$. This is a finite set of $(\mathfrak{g}, K_x)$-modules, all having the same infinitesimal character.

Define

$$\Pi(\phi) = \{[x, \pi] \mid x \in \mathcal{X}_1, \pi \in \Pi(x, \phi)\}. \tag{4.1}$$

We embed $\Pi(x, \phi)$ in $\Pi(\phi)$ by the map $\pi \to [x, \pi]$. This map is injective, since $[x, \pi] = [x, \pi']$ if and only if there exists $g \in G$ such that $gxg^{-1} = x$ and $\pi^g \simeq \pi'$. This holds if and only if $g \in K_x$ in which case $\pi^g \simeq \pi$.

It is helpful to make some choices to make this more explicit. Choose a set $\mathcal{X}_1'$ of representatives of $\mathcal{X}_1/W$; by (2.4)(b) $\mathcal{X}_1'$ parametrizes the strong real forms. Then there is a canonical bijection

$$\Pi(\phi) \stackrel{1-1}{\longleftrightarrow} \coprod_{x \in \mathcal{X}_1'} \Pi(x, \phi). \tag{4.2}$$

Each $\Pi(x, \phi)$ is an L-packet of $(\mathfrak{g}, K_x)$-modules. Suppose two strong real forms map to the same weak real form in (2.2)(a), i.e. $x, x' \in \mathcal{X}_1'$ satisfy: x' is conjugate to zx for some $z \in Z$. Then $K_x \simeq K_{x'}$, $\Pi(x, \phi)$ and $\Pi(x, \phi')$ are isomorphic, and this L-packet occurs twice.

Thus, unless G is adjoint these sets are often larger than necessary. (In particular if G is not semisimple they are infinite). For these reasons it is sometimes helpful to fix an element $z \in Z$, and define (cf. (2.6))

$$\Pi_z(\phi) = \{[x, \pi] \mid x^2 = z\} \subset \Pi(\phi). \tag{4.3}$$

These sets are finite, and the number of strong real forms mapping to a single weak real form is small, and often 1. On the other hand for a fixed z some real forms may fail to occur. For an example see Section 8.1.

5 Discrete Series L-packets

We specialize to the case of discrete series. Let $\phi : W_{\mathbb{R}} \to {}^{\vee}G^{\Gamma}$ be a discrete series parameter. These are characterized by any of several conditions, for example having finite centralizer. See [5, 10.3 and 10.5]. This determines an infinitesimal character, and $\Pi(x,\phi)$ is the set of discrete series $(\mathfrak{g}, K_x)$-modules with this infinitesimal character.

Here is a little more detail. Write $W_{\mathbb{R}} = \langle \mathbb{C}^*, j\rangle$ with relations $jzj^{-1} = \overline{z}$ and $j^2 = -1$. After conjugating by $G^\vee$ we may assume assume $\phi(\mathbb{C}^\times) \in H^\vee$. The fact that this is a discrete series parameter implies $\phi(j)h\phi(j)^{-1} = h^{-1}$ for $h \in H^\vee$, and $\phi(z) = (z/\overline{z})^\lambda \quad (z \in \mathbb{C}^*)$, with λ a regular element of $\rho + X^*(H)$ (where ρ is one-half the sum of any set of positive roots). We may assume λ is B-dominant. See [4, Section 7]. This implies

$$S_\phi = \{h \in H^\vee \mid h^2 = 1\}. \tag{5.1}$$

For $x \in \mathcal{X}_1$, $\gamma \in \rho + X^*(H)$ regular, write $\pi_x(\gamma)$ for the discrete series $(\mathfrak{g}, K_x)$-module with Harish-Chandra parameter γ. Then $\pi_x(\lambda) \simeq \pi_x(w\lambda)$ if and only if $w \in W(K_x)$, the Weyl group of H in K_x. Thus

$$\Pi(x,\phi) = \{\pi_x(w^{-1}\lambda) \mid w \in W/W(K_x)\}. \tag{5.2)(a}$$

This embeds in $\Pi(\phi)$ as

$$\Pi(x,\phi) = \{[x, \pi_x(w^{-1}\lambda)] \mid w \in W/W(K_x)\}. \tag{5.2)(b}$$

A key point is that we may identify this subset in a different way, using $[x, \pi_x(w^{-1}\lambda)] = w[x, \pi_x(w^{-1}\lambda)] = [wx, \pi_{wx}(\lambda)]$ for all $w \in W$, i.e.

$$\Pi(x,\phi) = \{[wx, \pi_{wx}(\lambda)] \mid w \in W/W(K_x)\}. \tag{5.2)(c}$$

Recall ϕ determines λ, so the second coordinate is determined by the first, so there is a bijection

$$\Pi(x,\phi) \xleftrightarrow{1-1} \{wx \mid w \in W/W(K_x)\} = \{y \in \mathcal{X}_1 \mid y \sim x\}. \tag{5.2)(d}$$

Thus we obtain a map

$$\mathcal{X}_1 \to \Pi(\phi), \quad x \to [x, \pi_x(\lambda)]. \tag{5.2)(e}$$

By the preceding discussion this is a bijection. This proves:

Proposition 5.3 *Suppose ϕ is a discrete series L-parameter for G. There is a canonical bijection*

$$\mathcal{X}_1 \overset{1-1}{\longleftrightarrow} \Pi(\phi). \tag{5.4}$$

Given $x \in \mathcal{X}_1$ this restricts to a bijection

$$\{y \in \mathcal{X}_1 \mid y \sim x\} \overset{1-1}{\longleftrightarrow} \Pi(x, \phi). \tag{5.5}$$

Given $z \in Z$ it restricts to a bijection (cf. (2.6) *and* (4.3)*)*

$$\mathcal{X}_1[z] \overset{1-1}{\longleftrightarrow} \Pi_z(\phi). \tag{5.6}$$

We say a $(\mathfrak{g}, K_x)$-module π is *generic* if the corresponding representation $\pi_{\mathbb{R}}$ of $G(\mathbb{R})$ admits a Whittaker model. This is equivalent to: π is *large*, i.e. has maximal Gelfand-Kirillov dimension. This implies $G(\mathbb{R})$ is quasisplit. See [13, Section 6].

It is important to know which elements of $\mathcal{X}_1$ correspond via (5.4) to generic discrete series representations. A discrete series representation $\pi(\lambda)$ is large if and only if every simple root in the chamber defined by λ is noncompact (for example see [13, Theorem 6.2(f)]). In our setting a root α is compact with respect to θ_x if $\alpha(x) = 1$, and is noncompact if $\alpha(x) = -1$. It is enough to consider simple roots, so:

Lemma 5.7 *In the bijection* (5.4) *$x \in \mathcal{X}_1$ corresponds to a generic discrete series representation if and only if*

$$\alpha(x) = -1 \quad \textit{for all simple roots } \alpha. \tag{5.8}$$

Let $\rho^\vee = \frac{1}{2}\sum \alpha^\vee$ where the sum is over the roots of $H^\vee$ in $B^\vee$. Then

$$x_b = \exp(\pi i \rho^\vee) \tag{5.9}$$

is a canonical element satisfying $\alpha(x_b) = (-1)^{\langle \alpha, \rho^\vee \rangle} = -1$ for all simple roots α. This provides a canonical basepoint in $\mathcal{X}_1$, corresponding to a generic discrete series representation of a quasisplit strong involution of G. This plays an important role in what follows.

6 The groups S_ϕ and $\widetilde{S_\phi}$

Fix a discrete series L-homomorphism ϕ. Let

$$S_\phi = \mathrm{Cent}_{G^\vee}(\mathrm{Image}(\phi)). \tag{6.1}$$

We want to relate characters of S_ϕ to elements of $\Pi(\phi)$.

Recall (5.1) $S_\phi = \{h \in H^\vee \mid h^2 = 1\}$, and by (1.3) this is isomorphic to $X_*(H^\vee)/2X_*(H^\vee) = X^*(H)/2X^*(H)$.

Let $p : G^\vee_{sc} \to G^\vee$ be the *topologically* simply connected cover of $G^\vee$. Thus $G^\vee_{sc} = \mathbb{C}^n \times G^\vee_{d,sc}$ where $n = \dim(Z)$ and $G^\vee_{d,sc}$ is the simply connected cover of the derived group $G^\vee_d$. Define

$$\widetilde{S}_\phi = p^{-1}(S_\phi) \subset G^\vee_{sc}. \tag{6.2}$$

This is a variant of the *algebraic* cover $G^{\vee,alg}$ of [3, (1.16) and (5.10)], which is the projective limit of the finite covers of $G^\vee$. The difference is due to the fact that we define strong involutions to be elliptic elements, rather than having finite order (Definition 2.1).

The inverse image of $H^\vee$ in $G^\vee_{sc}$ is isomorphic to $X_*(H^\vee)\otimes\mathbb{C}/R^\vee(G^\vee, H^\vee) \simeq X^*(H) \otimes \mathbb{C}/R(G, H)$. From this it follows easily that there is a natural isomorphism

$$X^*(H)/2R \simeq \widetilde{S}_\phi. \tag{6.3}$$

This contains a lattice of rank $\dim(Z)$. For $\lambda \in X^*(H)$ write $s(\lambda)$ for the corresponding element of $\widetilde{S_\phi}$. The map $p : \widetilde{S_\phi} \to S_\phi$ is then the natural map

$$\widetilde{S_\phi} \simeq X^*(H)/2R \to X^*(H)/2X^*(H) \simeq S_\phi. \tag{6.4}$$

Use $\widehat{\ }$ to denote Pontriagin dual.

Lemma 6.5 *There is a canonical group isomorphism $\mathcal{X}_1 \simeq \widetilde{S_\phi}\widehat{\ }$.*

Proof. Consider the restriction map from $X^*(H)$ to the Pontriagin dual of $\mathcal{X}_1$. This is surjective, since any character of the maximal compact subgroup H_c of H extends to an algebraic character of H, and $\mathcal{X}_1 \subset H_c$ by definition. The kernel is $2R$, so $\widehat{\mathcal{X}_1} \simeq X^*(H)/2R \simeq \widetilde{S_\phi}$ by (6.3). Apply Pontriagin duality. ■

Explicitly, using the map s of (6.3), the character χ of $\widetilde{S_\phi}$ corresponding to $x \in \mathcal{X}_1$ is given by

$$\chi(s(\lambda)) = \lambda(x) \quad (\lambda \in X^*(H)). \tag{6.6}$$

Together with (5.4) we obtain bijections $\widetilde{S_\phi}\,\hat{} \overset{1-1}{\longleftrightarrow} \mathcal{X}_1 \overset{1-1}{\longleftrightarrow} \Pi(\phi)$, taking the trivial character of $\widetilde{S_\phi}$ to the identity element of $\mathcal{X}_1$, and hence to a finite dimensional representation of a compact strong real form G (cf. Example 2.8). This is not the right normalization: we prefer the basepoint to be a generic discrete series of a quasisplit strong involution. This is provided (canonically) by the element x_b of 5.9.

Definition 6.7 *For $x \in \mathcal{X}_1$ define $\tau_x \in \widetilde{S_\phi}\,\hat{}$ by*

$$\tau_x(s(\lambda)) = \lambda(x x_b^{-1}). \tag{6.8}$$

In other words if $x = \exp(\pi i \gamma^\vee)$ ($\gamma^\vee \in X_*(H)$) then

$$\tau_x(s(\lambda)) = e^{\pi i \langle \gamma^\vee - \rho^\vee, \lambda \rangle}. \tag{6.9}$$

This defines a canonical bijection of sets

$$\widetilde{S_\phi}\,\hat{} \overset{1-1}{\longleftrightarrow} \mathcal{X}_1 \tag{6.10}$$

taking the trivial character to x_b. Composing with (5.4) we obtain:

Proposition 6.11 *There is a a canonical bijection*

$$\widetilde{S_\phi}\,\hat{} \overset{1-1}{\longleftrightarrow} \Pi(\phi). \tag{6.12}$$

taking the trivial character of $\widetilde{S_\phi}$ to $[x_b, \pi_{x_b}(\lambda)]$, a generic discrete series representation of a quasisplit strong real form of G.

Definition 6.13 *Given $\tau \in \widetilde{S_\phi}\,\hat{}$ let $x_\tau \in \mathcal{X}_1$ be the corresponding element via (6.10), and let $\pi(\tau) = [x_\tau, \pi_{x_\tau}(\lambda)] \in \Pi(\phi)$ be the corresponding element of $\Pi(\phi)$ via (6.12).*

This bijection has the advantage that it is canonical. On the other hand it differs from more familiar constructions in several respects. It involves the group $\widetilde{S_\phi}$; in applications (and over other fields) it is the more natural group

S_ϕ which plays a role. Furthermore $\widetilde{S_\phi}$ is often larger than necessary (even infinite), due in part to the failure of injectivity in the map (2.2)(b).

In any event for $z \in Z$ it is helpful to identify the image of $\Pi_z(\phi)$ under the bijection (6.12). The basic issue is to identify $\widehat{S_\phi}$ as a subset of $\widetilde{S_\phi}\widehat{\ }$:

$$\widehat{S_\phi} = \{\tau_{xx_b} \mid x^2 = 1\} = \{\tau_x \mid x^2 = z(\rho^\vee)\} \overset{1-1}{\longleftrightarrow} \Pi_{z(\rho^\vee)}(\phi). \tag{6.14}$$

The last set is the representations of pure strong real forms.

Now fix $y \in \mathcal{X}_1[z]$. Then $\tau_y\widehat{S_\phi} = \{\tau_y\tau_x \mid x^2 = z(\rho^\vee)\}$. Using the elementary identity $\tau_y\tau_x = \tau_{yxx_b^{-1}}$, and noting that $(yxx_b^{-1})^2 = y^2 = z$, we have the following result.

Lemma 6.15 *Fix $z \in Z$ and choose $y \in \mathcal{X}_1[z]$. Then $\tau_y\widehat{S_\phi} \subset \widetilde{S_\phi}\widehat{\ }$ is independent of the choice of y, and the bijection of Proposition 6.11 restricts to a canonical bijection*

$$\tau_y\widehat{S_\phi} \overset{1-1}{\longleftrightarrow} \Pi_z(\phi). \tag{6.16)(a}$$

In particular taking $y = x_b$ we obtain a canonical bijection

$$\widehat{S_\phi} \overset{1-1}{\longleftrightarrow} \Pi_{z(\rho^\vee)}(\phi) \tag{6.16)(b}$$

taking the identity to $[x_b, \pi_{x_b}(\lambda)]$, a generic discrete series representation of a quasisplit strong real form of G.

Although Proposition 6.11 is not perfectly suited to the classical theory, neither is the Lemma: the map from strong real forms of type z to weak real forms is not necessarily surjective or injective. Also note that, except in the setting of (6.16)(b), a further choice (of y) is required to give a bijection $\widehat{S_\phi} \overset{1-1}{\longleftrightarrow} \Pi_z(\phi)$. See Section 9.

7 Endoscopic Lifting

We continue in the setting of the previous Section, with a discrete series parameter ϕ, and the bijections $\widetilde{S_\phi}\widehat{\ } \ni \tau \to x_\tau \in \mathcal{X}_1$ and $\pi(\tau) \in \Pi(\phi)$ of Definition 6.13.

For $x \in \mathcal{X}_1$ let

$$e(x) = (-1)^{\dim(G/K_x) - \dim(G/K_{x_b})}. \tag{7.1}$$

Recall (5.9) K_{x_b} is the complexified maximal compact of the quasisplit real form of G; $e(x)$ is the *Kottwitz invariant* [6] of a real form of G defined by x.

Definition 7.2 *Fix $\tilde{s} \in \widetilde{S_\phi}$. Let*

$$\text{(7.3)(a)} \qquad \widetilde{\eta}(\tilde{s}) = \sum_{\tau \in \widetilde{S_\phi}^\wedge} e(x_\tau)\tau(\tilde{s})\pi(\tau).$$

Fix $x \in \mathcal{X}_1$ and let

$$\text{(7.3)(b)} \qquad \widetilde{\eta}_x(\tilde{s}) = e(x) \sum_{\{\tau \,|\, x_\tau \sim x\}} \tau(\tilde{s})\pi(\tau).$$

This only depends on the image of x in $\mathcal{X}_1/W$, and is a linear combination of representations of this strong real form of G (cf. (2.4)(b)*).*

This is a special case of the discussion on pages 20-21 of [3]; see [3, Theorem 26.8] for details. This is a collection of virtual representations of strong real forms of G, parametrized by $\widetilde{S_\phi}$. Now [3, Theorem 1.39] relates these virtual characters to characters of an endoscopic group, and is the version of endoscopic lifting of [3]. For the purposes of these notes we use only Definition 7.2.

In order to compare this with the lifting defined by Shelstad we write (7.3)(b) in more familiar terms. First of all we may write

$$\text{(7.4)} \qquad \widetilde{\eta}_x(\tilde{s}) = e(x)\tau_x(\tilde{s}) \sum_{\{\tau \,|\, x_\tau \sim x\}} (\tau(\tilde{s})/\tau_x(\tilde{s}))\pi(\tau).$$

Note that $\{\tau \,|\, x_\tau \sim x\} = \{\tau_{wx} \,|\, w \in W/W(K_x)\}$. Recall $\pi(\tau_{wx}) = [wx, \pi_{wx}(\lambda)]$, and (cf. 5.2) this equals $[x, \pi_x(w^{-1}\lambda)]$, where $\pi_x(w^{-1}\lambda)$ is the discrete series $(\mathfrak{g}, K_x)$-module with Harish-Chandra parameter $w^{-1}\lambda$. We will see momentarily that τ_{wx}/τ_x factors to S_ϕ, so for $x \in \mathcal{X}_1, s \in S_\phi$ define a function $\kappa(x,s)$ on W:

$$\text{(7.5)} \qquad \kappa(x,s)(w) = (\tau_{wx}/\tau_x)(s).$$

Then:

Proposition 7.6 *Given $x \in \mathcal{X}_1, \tilde{s} \in \widetilde{S_\phi}$, let $s = p(\tilde{s}) \in S_\phi$. Then*

$$\text{(7.7)} \qquad \widetilde{\eta}_x(\tilde{s}) = e(x)\tau_x(\tilde{s}) \sum_{w \in W/W(K_x)} \kappa(x,s)(w)\pi_x(w^{-1}\lambda).$$

Here are some elementary properties of the constants $\kappa(x,s)$.

Lemma 7.8 *(1)* $\kappa(x,s): W \to \pm 1$*;*

(2) $\kappa(x,s)(w) = 1$ *for* $w \in W(K_x)$*;*

(3) $\kappa(x,s)(w_1 w_2) = \kappa(x,s)(w_1)$ *for* $w_2 \in W(K_x)$*;*

(4) $\kappa(x,s)(w_1 w_2) = \kappa(w_2 x, s)(w_1)\kappa(x,s)(w_2)$*;*

Proof. If $x = \exp(\pi i \gamma^\vee)$ ($\gamma^\vee \in X_*(H^\vee)$) then

(7.9) $$(\tau_{wx}/\tau_x)(s(\lambda)) = e^{\pi i \langle w\gamma^\vee - \gamma^\vee, \lambda\rangle} \quad (\lambda \in X^*(H)).$$

This equals 1 if $\lambda \in 2X_*(H)$. Since the kernel of $\widetilde{S_\phi} \to S_\phi$ is $2X_*(H^\vee)/2R(H^\vee)$ (6.3) $\kappa(x,s)$ is well defined for $s \in S_\phi$. It is easy to see (7.9) implies (1) and (4), and (2) and (3) are special cases of (4). ■

The functions $\kappa(x,s)$ appear in [12, Section 3]. It follows from Lemma 7.8 that $\widetilde{\eta}_x(\widetilde{s})$ agrees, up to a constant of absolute value 1 with the lifting of Shelstad defined in [10, Section 4]. This is also discussed in [3, pg. 289]. See [11] for an update of [10], using the transfer factors of Langlands and Shelstad [9], which makes some use of a cover of S_ϕ and multiple real forms.

Definition 7.2 is entirely canonical, but involves the group $\widetilde{S_\phi}$. The part of the sum (7.3)(a) over $\widehat{S_\phi} \subset \widetilde{S_\phi}\widehat{\ }$ involves just the pure strong real forms. Thus define

(7.10)(a) $$\eta_{z(\rho^\vee)}(s) = \sum_{\tau \in \widehat{S_\phi}} e(x_\tau)\tau(s)\pi(\tau).$$

This is a linear combination of the representations of pure strong real forms.

More generally, (cf. (6.14) and (6.16)(b)), for $z \in Z$, choose $x \in \mathcal{X}_1[z]$. The part of the sum (7.3)(a) giving a linear combination of the elements of $\Pi_z(\phi)$ is:

(7.10)(b) $$\eta_z(\tilde{s}) = \tau_x(\tilde{s}) \sum_{\tau \in \widehat{S_\phi}} e(x_{\tau\tau_x})\tau(s)\pi(\tau\tau_x)$$

where $s = p(\tilde{s}) \in S_\phi$. Note that (a) is obtained from (b) by making the canonical choice $x = x_b$ (so $\tau_x = 1$); in general there is no canonical choice of a basepoint x.

We return to (7.3)(b). Note that $x_\tau \sim x$ implies $\tau \in \tau_x\widehat{S_\phi}$ (cf. (6.16)(a)), so we can write the sum (7.3)(b) as being over $\{\tau\tau_x \mid \tau \in \widehat{S_\phi}, x_{\tau\tau_x} \sim x\}$. So: given $x \in \mathcal{X}_1, s \in S_\phi$ and $\epsilon = \pm 1$ let

$$\widehat{S_\phi}(x,s)_\epsilon = \{\tau \in \widehat{S_\phi} \mid x_{\tau\tau_x} \sim x, \tau(s) = \epsilon\}. \tag{7.11}$$

Then the sum (7.3)(b) becomes

$$\widetilde{\eta}_x(\tilde{s}) = e(x)\tau_x(\tilde{s})[\sum_{\widehat{S_\phi}(x,s)_+} \pi(\tau\tau_x) - \sum_{\widehat{S_\phi}(x,s)_-} \pi(\tau\tau_x)]. \tag{7.12}$$

The right hand side is a linear combination of representations of the strong real form x; the coefficients are signs times $\tau_x(\tilde{s})$, which is a complex number of absolute value 1. (If we chose our strong involutions to have finite order as in [3] this would be a root of unity.) The question of the signs, and precisely how they depend on the choices involved, was brought to our attention by Michael Harris; see ([7], pages 200-201). The case of $U(p,q)$ is discussed in Section 10.

8 Example: Strong Real forms of GL$(n,\mathbb{C})$

Let (,) be a non-degenerate Hermitian form on $\mathbb{C}^n$. It is determined up to equivalence by its signature (p,q) with $p+q=n$, and the symmetry group of this form is denoted $U(p,q)$. These are the weak real forms of GL$(n,\mathbb{C})$ of equal rank.

The map from $\mathcal{X}_1$ to weak real forms is surjective, with infinite fibers. For example

$$x = \mathrm{diag}(\overbrace{\alpha,\dots,\alpha}^{p}, \overbrace{-\alpha,\dots,-\alpha}^{q}) \tag{8.1}$$

maps to $U(p,q)$ for any $\alpha \in S^1$. It is much better to fix $z \in Z$ and consider $\mathcal{X}_1[z]$, which is finite. Note that $Z/Z^2 = 1$, so $\mathcal{X}_1[z]$ is the reduced parameter space $\mathcal{X}_1^r$ (2.7).

The Hermitian form $-(,)$ is not equivalent to (,) (unless $p=q$) but their symmetry groups $U(p,q)$ and $U(q,p)$ are isomorphic. Thus the map from equivalence classes of Hermitian forms to weak real forms is two-to-one (unless $p=q$). It turns out we can identify strong real forms of type z with equivalence classes of Hermitian forms (although not canonically). Thus we can think of strong real forms of type z as $\{U(p,q)\,|\,p+q=n\}$ where we *distinguish* $U(p,q)$ and $U(q,p)$. We proceed to make this precise.

Let $G = \mathrm{GL}(n,\mathbb{C})$ and fix $z = \beta I \in Z$ with $|\beta| = 1$. Furthermore fix α with $\alpha^2 = 1$. Then

$$\mathcal{X}_1[z] = \{\alpha \mathrm{diag}(\pm 1, \dots, \pm 1)\}. \tag{8.2}$$

The elements conjugate to

$$x = \mathrm{diag}(\overbrace{\alpha, \dots, \alpha}^{p}, \overbrace{-\alpha, \dots, -\alpha}^{q}) \tag{8.3}$$

constitute a single strong real form, which we think of as $U(p,q)$. Those conjugate to $-x$ also constitute a single strong real form, distinct from the previous one unless $p = q$, which we think of as $U(q,p)$. Note that this labelling depends on a choice of $\sqrt{\beta}$. In any event the elements conjugate to $\pm x$ all map to the same weak real form $U(p,q) = U(q,p)$.

The pure strong real forms are an important special case. Note that $z(\rho^\vee) = (-1)^{n-1}I$, so $\beta = (-1)^{n-1}$ and we can take $\alpha = i^{n-1}$. Thus with this choice $x \in \mathcal{X}_1[z(\rho^\vee)]$ gives the strong real form $U(p,q)$ where p is the dimension of the i^{n-1} eigenspace of x.

8.1 Strong real forms of SL$(n,\mathbb{C})$

The equal rank real forms of $\mathrm{SL}(n,\mathbb{C})$ are the special unitary groups $SU(p,q)$ with $p+q=n$.

If n is odd then the map from strong real forms to weak real forms is bijective. If n is even the picture is essentially the same as for $\mathrm{GL}(n,\mathbb{C})$. In both cases we work with the reduced parameter space. Here are the details.

First assume n is odd. Since $Z/Z^2 = 1$ we can choose arbitrary $z \in Z$ and take the reduced parameter space $\mathcal{X}_1^r$ to be $\mathcal{X}_1[z]$ (cf. 2.7), and the map from strong real forms of type z to weak real forms is surjective. In fact it is a bijection in this case. For example take $z = z(\rho^\vee) = I$. Then $\mathcal{X}_1[I] \ni x = \mathrm{diag}(\pm 1, \dots, \pm 1)$ maps to the real form $SU(p,q)$ where q is the number of minus signs, which is even. For example if $n = 5$ the weak real forms are $SU(5,0), SU(3,2)$ and $SU(1,4)$.

Now suppose $n = 2m$ is even. In this case $z(\rho^\vee) = -I$, and the pure strong involutions are

$$\mathcal{X}_1[z(\rho^\vee)] = \{i\mathrm{diag}(\epsilon_1, \dots, \epsilon_n)\} \tag{8.4}$$

with $\epsilon_i = \pm 1$ and $\prod \epsilon_i = (-1)^m$.

As in (8.3) the elements conjugate to

$$x = \mathrm{diag}(\overbrace{i, \dots, i}^{p}, \overbrace{-i, \dots, -i}^{q}) \tag{8.5}$$

map to the weak real form $SU(p,q)$, where now $p \equiv q \equiv m \pmod 2$. The map from pure strong real forms to weak real forms is not surjective.

Since $n = 2m$ is even $|Z/Z^2| = 2$, so we choose another element of Z. Take

$$z' = \begin{cases} I & m \text{ is odd} \\ e^{2\pi i/n} I & m \text{ is even.} \end{cases} \tag{8.6}$$

Then

$$\mathcal{X}_1[z'] = \{\mathrm{diag}(\epsilon_1, \dots, \epsilon_n) \mid \prod \epsilon_i = 1\} \quad (m \text{ odd}) \tag{8.7}$$

or

$$\mathcal{X}_1[z'] = \{e^{\pi i/n}\mathrm{diag}(\epsilon_1, \dots, \epsilon_n) \mid \prod \epsilon_i = -1\} \quad (m \text{ even}). \tag{8.8}$$

There is a two-to-one map from strong real forms of type z' to the weak real forms $SU(2p, 2q)$ (m odd) or $SU(2p+1, 2q+1)$ (m even).

Thus we see the map from strong real forms of type $z(\rho^\vee) = -I$ or z' to weak real forms two-to-one, except the fiber of the weak real form $SU(m,m)$ is a singleton. This is just like the case of $U(p,q)$ except that Z/Z^2 now has two elements.R

For example if $n = 6$, the pure real forms are $SU(5,1), SU(3,3)$ and $SU(1,5)$, with $z = z(\rho^\vee) = -I$. Taking $z' = I$ gives the other strong real forms $SU(6,0), SU(4,2), SU(2,4)$ and $SU(0,6)$.

Take $n = 8$. In this case the pure real forms, given by $z(\rho^\vee) = -I$, are $SU(8,0), SU(6,2), SU(4,4), SU(2,6)$ and $SU(0,8)$. On the other hand $z' = e^{2\pi i/8}$ gives the strong real forms $SU(7,1), SU(5,3), SU(3,5)$ and $SU(1,7)$.

9 Discrete series of $U(p,q)$

Let $G = \mathrm{GL}(n, \mathbb{C})$ and suppose ϕ is a discrete series L-parameter. Fix $z \in Z$ and $y \in \mathcal{X}_1[z]$. We make the bijection $\tau_y \widehat{S_\phi} \xleftrightarrow{1-1} \Pi_z(\phi)$ of (6.16)(a) explicit.

Fix $z = \beta I$ for some $\beta \in \mathbb{C}^\times$. In this case there is a natural choice of basepoint y (or rather two such). Recall $x_b = i^{n-1}(1, -1, \dots, (-1)^{n-1}) \in \mathcal{X}[z(\rho^\vee)]$ defines a generic discrete series representation of a quasisplit strong real form. Choose $\alpha^2 = \beta$ and let

$$y = \alpha i^{-n+1} x_b = \alpha(1, -1, \dots, (-1)^{n-1}) \in X_1[z]. \tag{9.1}$$

We have $S_\phi \simeq \mathbb{Z}/2\mathbb{Z}^n$, embded diagaonlly in H. For $\delta_i = \pm 1$ write $\{\delta_1, \dots, \delta_n\}$ for the corresponding character of S_ϕ.

Lemma 9.2 *With y as in* (9.1) *the bijection $\tau_y \widehat{S_\phi} \xleftrightarrow{1-1} \mathcal{X}_1[z]$ of* (6.16)(a) *is given as follows. If $\tau = \{\delta_1, \dots, \delta_n\}$ then*

$$\tau_y \tau \to \alpha(\delta_1, -\delta_2, \delta_3, \dots, (-1)^{n-1}\delta_n). \tag{9.3}$$

In particular the case of pure strong real forms is given by $z = (-1)^{n-1} I$, so it is natural to take $\alpha = i^{n-1}$, so $y = x_b$ and $\tau_y = 1$. Then the bijection is

$$\widehat{S_\phi} \ni \tau = \{\delta_1, \dots, \delta_n\} \to i^{n-1}(\delta_1, -\delta_2, \dots, (-1)^{n-1}\delta_n) \in \mathcal{X}_1[z(\rho^\vee)]. \tag{9.4}$$

This is an easy exercise in the definitions.

Let $\tau_0 = \{1, -1, \dots, (-1)^n\} \in \widehat{S_\phi}$. If $\tau = \{\delta_1, \dots, \delta_n\}$ let $p(\tau), q(\tau)$ be the number of δ_i equal to $1, -1$, respectively.

Lemma 9.5 *In the setting of the Lemma suppose $\tau, \tau' \in \widehat{S_\phi}$ and $\tau_y \tau \to x, \tau_y \tau' \to x'$ via the bijection* (9.3).

Then x, x' define the same strong real form if and only if $p(\tau\tau_0) = p(\tau'\tau_0)$. They map to the same weak real form if and only if $p(\tau\tau_0) = p(\tau'\tau_0)$ or $p(\tau\tau_0) = q(\tau'\tau_0)$.

As discussed in Section 8 the choice of $\alpha = \sqrt{\beta}$ amounts to a labelling of the strong real forms as $U(p, q)$ or $U(q, p)$. That is, we may *define* the strong real form $U(p, q)$ to be those $x \in \mathcal{X}_1[z]$ for which the dimension of the α-eigenspace is p. With this convention we have:

Corollary 9.6 *$\tau_y \tau \in \tau_y \widehat{S_\phi}$ goes to a discrete series representation of the strong real form $U(p(\tau\tau_0), q(\tau\tau_0))$.*

The most natural case is $z = z(\rho^\vee)$, i.e. $\beta = (-1)^{n-1}$, in which case it is natural to take $\alpha = i^{n-1}$. Then $U(p, q)$ is the strong real form corresponding to those x with i^{n-1} occuring as an eigenvalue of multiplicity p.

Note that each weak real form $U(p,q)$ occurs twice (as the strong real forms $U(p,q)$ and $U(q,p)$) unless $p = q$. The total number of discrete series representations is therefore $\sum_{p=0}^{n}\binom{n}{p} = 2^n = |\widehat{S_\phi}|$.

Note that $x \to -x$ is an automorphism of $\mathcal{X}_1[z]$, taking the strong real form $U(p,q)$ to $U(q,p)$. It is helpful to consider the affect of this automorphism on our basepoints, i.e. generic discrete series representations. We consider the pure case $z = z(\rho^\vee)$, others are similar.

It is not hard to see that for $\tau \in \widehat{S_\phi}$, $\pi(\tau)$ is generic if and only if $x_\tau = \pm x_b$. If $n = 2m$, $\pm x_b$ are conjugate, so we're getting two generic discrete series representations of the same strong real form $U(m,m)$. If $n = 2m+1$ then $\pm x_b$ are not conjugate; these correspond to generic discrete series representations, one each of the two strong real forms $U(m+1,m)$ and $U(m,m+1)$.

It is helpful to express the bijection (5.4) in more familiar parameters. This is primarily an exercise in the definitions. We illustrate this with the examples of GL(4) and GL(3).

9.1 Example: GL(4)

Consider pure strong real forms of $\mathrm{GL}(4,\mathbb{C})$, so take $z = -I$. In the setting of (9.4) we have $\alpha = -i$ and $y = x_b = -i\mathrm{diag}(1,-1,1,-1)$, We work at infinitesimal character $(4,3,2,1)$, which is a central shift of $\rho = (3/2,1/2,-1/2,-3/2)$.

The quasisplit group is $U(2,2)$. Write a Harish-Chandra parameter for $U(2,2)$ as $(a,b;c,d)$. This indicates that the positive compact roots are $e_1 - e_2, e_3 - e_4$ in the usual notation, and $e_2 - e_3$ is noncompact. Write $\pi(a,b;c,d)$ for the discrete series representation with this Harish-Chandra parameter. Then $\pi(4,2;3,1)$ is a generic discrete series representation: the simple roots are $e_1 - e_3, e_3 - e_2, e_2 - e_4$, all of which are noncompact. The other generic discrete series representation with this infinitesimal character is $\pi(3,1;4,2)$.

We need to fix a generic discrete series representation of $U(2,2)$: choose $\pi(4,2;3,1)$. We then easily compute the following table, showing the bijections $\widehat{S_\phi} \to \mathcal{X}_1[z(\rho^\vee)]$ of (9.4) and $\widehat{S_\phi} \to \Pi_{z(\rho^\vee)}(\phi)$ of (6.16)(b).

Recall labelling the strong real forms as $U(p,q), U(q,p)$ requires a choice of $\sqrt{-1}$, which it was most natural to take to be $i^{n-1} = -i$. Therefore *define* the strong real form $U(p,q)$ to be the conjugacy class of x with eigenvalue $-i$ of multiplicity p.

$U(2,2)$

$\tau \in \widehat{S_\phi}$	$x \in \mathcal{X}_1[z(\rho^\vee)]$	HC parameter
{1,1,1,1}	-i(1,-1,1,-1)	(4,2;3,1)
{1,-1,-1,1}	-i(1,1,-1,-1)	(4,3;2,1)
{1,1,-1,-1}	-i(1,-1,-1,1)	(4,1;3,2)
{-1,-1,1,1}	-i(-1,1,1,-1)	(3,1;4,2)
{-1,-1,-1,-1}	-i(-1,1,-1,1)	(3,2;4,1)
{-1,1,1,-1}	-i(-1,-1,1,1)	(2,1;4,3)

The representations of the strong real form $U(3,1)$ are:

$U(3,1)$

$\tau \in \widehat{S_\phi}$	$x \in \mathcal{X}_1[z(\rho^\vee)]$	HC parameter
$\{1,-1,1,1\}$	-i(1,1,1,-1)	(4,3,2;1)
$\{1,-1,-1,-1\}$	-i(1,1,-1,1)	(4,3,1;2)
$\{1,1,1,-1\}$	-i(1,-1,1,1)	(4,2,1;3)
$\{-1,-1,1,-1\}$	-i(-1,1,1,1)	(3,2,1;4)

The negatives of these three parameters give representations of the strong real form $U(1,3)$:

$U(1,3)$

$\tau \in \widehat{S_\phi}$	$x \in \mathcal{X}_1[z(\rho^\vee)]$	HC parameter
$\{-1,1,-1,-1\}$	-i(-1,-1,-1,1)	(1;4,3,2)
$\{-1,1,1,1\}$	-i(-1,-1,1,-1)	(2;4,3,1)
$\{-1,-1,-1,1\}$	-i(-1,1,-1,-1)	(3;4,3,1)
$\{1,1,-1,1\}$	-i(1,-1,-1,-1)	(4;3,2,1)

Finally for the compact strong real forms we have:

$\tau \in \widehat{S_\phi}$	$x \in \mathcal{X}_1[z(\rho^\vee)]$	G	HC parameter
$\{1,-1,1,-1\}$	-i(1,1,1,1)	$U(4,0)$	(4,3,2,1)
$\{-1,1,-1,1\}$	-i(-1,-1,-1,-1)	$U(0,4)$	(4,3,2,1)

As discussed above this entire table has an automorphism given by multiplication by -1, which has the effect of switching $U(p,q)$ and $U(q,p)$, and interchanging the two generic discrete series representations of $U(2,2)$.

9.2 Example: GL(3)

This is a little different from the preceding example since 3 is odd. In this case $x_b = -(1,-1,1)$. Fix infinitesimal character $(3,2,1)$. In this case there is only one generic discrete series representation of $U(2,1)$, so there is no choice to make.

The result is:

$\tau \in \widehat{S_\phi}$	$x \in \mathcal{X}_1[z(\rho^\vee)]$	G	HC parameter
{1,1,1}	-(1,-1,1)	$U(2,1)$	(3,1;2)
{1,-1,-1}	-(1,1,-1)	$U(2,1)$	(3,2;1)
{-1,-1,1}	-(-1,1,1)	$U(2,1)$	(2,1;3)
{1,-1,1}	-(1,1,1)	$U(3,0)$	(3,2,1)
{-1,-1,-1}	-(-1,1,-1)	$U(1,2)$	(2;3,1)
{-1,1,1}	-(-1,-1,1)	$U(1,2)$	(1;3,2)
{1,1,-1}	-(1,-1,-1)	$U(1,2)$	(3;2,1)
{-1,1,-1}	-(-1,-1,-1)	$U(0,3)$	(3,2,1)

In this case the two generic discrete representations are $-(1,-1,1)$ of $U(2,1)$ and $-(-1,1,-1)$ of $U(1,2)$.

10 Endoscopy for $U(p,q)$

We consider the question at the end of Section 7 in the case of $U(p,q)$. Set $n = p+q$ and $G = \mathrm{GL}(n,\mathbb{C})$.

Recall (7.11): for $x \in \mathcal{X}_1$, $s \in S_\phi$ and $\epsilon = \pm$ we set

$$\widehat{S_\phi}(x,s)_\epsilon = \{\tau \in \widehat{S_\phi} \mid x_{\tau\tau_x} \sim x,\ \tau(s) = \epsilon\}. \tag{10.1}$$

Let V be the standard module of $\mathrm{GL}(n,\mathbb{C})$, with basis $\{v_1,\dots,v_n\}$. Then $S_\phi \subset H^\vee \subset \mathrm{GL}(V)$ acts on V and its exterior algebra, write this action $s : \gamma \to s \cdot \gamma$. Write $S_\phi \simeq (\mathbb{Z}/2\mathbb{Z})^n$ so that $(s_1,\dots,s_n)\cdot v_i = s_i v_i$.

Proposition 10.2 *Given x, s and ϵ, choose α such that $x^2 = \alpha^2 I$. Let*

$$\delta = (\tau_x/\tau_{\alpha I})(s)\epsilon = \pm 1. \tag{10.3}$$

Let r be the dimension of the α-eigenspace of x. Then there is a natural bijection between $\widehat{S_\phi}(x,s)_\epsilon$ and a basis of the δ-eigenspace of s acting on $\Lambda^{n-r}(V)$.

Remark 10.4 Both δ and r depend on α, so write δ_α, r_α. The δ_α-eigenspace of s acting on $\Lambda^{n-r_\alpha}(V)$ is in bijection with the $\delta_\alpha \det(s)$-eigenspace of s acting on $\Lambda^{n-(n-r_\alpha)}(V)$. It is easy to see $\delta_\alpha \det(s) = \delta_{-\alpha}$ and $r_\alpha = n - r_{-\alpha}$, confirming that the statement is independent of the choice of α.

For an explicit description of the sign $(\tau_x/\tau_{\alpha I})(s)$ see (10.11).

Lemma 10.5 *Fix* $0 \le r \le n$. *Let* $S = (\mathbb{Z}/2\mathbb{Z})^n$, *embedded diagonally in* $GL(V) = GL(n, \mathbb{C})$. *Let* $\tau = diag(\delta_1, \dots, \delta_n)$ $(\delta_i = \pm 1)$ *be a character of* S, *and let* $q(\tau)$ *be the number of* δ_i *equal to* -1. *Suppose* $s \in S$ *and* $\epsilon = \pm 1$.

There is a natural bijection between a basis of

(10.6)(a) $$\{\gamma \in \Lambda^r(V) \mid s \cdot \gamma = \epsilon\gamma\}$$

and

(10.6)(b) $$\{\tau \in \widehat{S} \mid q(\tau) = r,\, \tau(s) = \epsilon\}$$

Proof. The proof is elementary. Suppose $1 \le i_1 < \dots < i_r \le n$ are the indices for which $\delta_{i_j} = -1$. Then $\tau(s) = \prod_j s_{i_j}$. Map τ to $\gamma = v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_r}$. Then $s \cdot \gamma = \prod_j s_{i_j}\gamma$. It is easy to see this is a bijection. ■

Proof of the Proposition. The fact that $x^2 = \alpha^2 I$ implies $x = \alpha(x_1, \dots, x_n)$ with $x_i \in \pm 1$. Two such elements are conjugate if and only if the dimension of their α-eigenspaces agree. Thus

(10.7) $$\widehat{S_\phi}(x,s)_\epsilon = \{\tau \in \widehat{S_\phi} \mid \text{dim of the } \alpha\text{-eigenspace of } x_{\tau\tau_x} = r, \tau(s) = \epsilon\}.$$

With notation as in Section 9 write $\tau = \{\delta_1, \dots, \delta_n\}$. A short calculation shows that

(10.8) $$x_{\tau\tau_x} = \alpha(\delta_1 x_1, \dots, \delta_n x_n).$$

Therefore if we let $\mu_x = \{x_1, \dots, x_n\} \in \widehat{S_\phi}$ then

(10.9) $$\widehat{S_\phi}(x,s)_\epsilon = \{\tau \in \widehat{S_\phi} \mid p(\tau\mu_x) = r, \tau(s) = \epsilon\}$$

where $p(*)$ is the number of times 1 occurs. Let $\tau' = \tau\mu_x$, so

(10.10) $$\widehat{S_\phi}(x,s)_\epsilon = \{\tau'\mu_x \mid \tau' \in \widehat{S_\phi},\, p(\tau') = r,\, \tau'(s) = \mu_x(s)\epsilon\}.$$

Replace the condition $p(\tau') = r$ with $q(\tau') = n - r$. It is straightforward to see that $\mu_x = \tau_x/\tau_{\alpha I}$. The result follows from the Lemma. ■

From the proof we see that if we write $x = \alpha(x_1, \dots, x_n)$ with $x_i \in \pm 1$, and $s = \mathrm{diag}(s_1, \dots, s_n)$ then

$$(\tau_x/\tau_{\alpha I})(s) = \prod_{\{i \,|\, x_i = -1\}} s_i \tag{10.11}$$

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