COMPUTING THE UNITARY DUAL

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1. Introduction

In the 1930s I. M. Gelfand outlined a program of abstract harmonic analysis, which offered a paradigm for the use of symmetry to study a very wide class of mathematical problems. A key technical step in Gelfand's program is

Problem 1.1. For every locally compact group G, determine the set $\widehat{G}_u$ of irreducible unitary representations G.

(Unitary representations, and the origins of this problem, will be explained in more detail in section 2 below.) The groups G that appear are the symmetry groups of the original problem. In problems of differential geometry, they are generally Lie groups. In problems of number theory, they may be algebraic groups over local fields. Because of these examples (and because of the work of [Ma] and [Du]) Gelfand's general question may in many important cases be reduced to

Problem 1.2. For every reductive group $\boldsymbol{G}$ defined over a local field F, determine the set $\widehat{\boldsymbol{G}(F)}_u$ of irreducible unitary representations $\boldsymbol{G}(F)$.

Examples of groups considered here are the general linear group $GL(n, F)$, the symplectic group $Sp(2n, F)$, orthogonal groups $SO(n, F)$, or the exceptional groups of type E_6, E_7, E_8, F_4 or G_2.

We propose to assemble a collection of mathematical and computational tools to address Problem 1.2. These tools may ultimately be able to solve Problem 1.2 completely for exceptional real Lie groups; and to support and guide the mathematical work that will be required to solve the problem for classical groups over any local field and exceptional groups over p-adic fields.

In Section 3 we will sketch a proof of:

Theorem 1.3. *Suppose G is the group of real points of a connected reductive algebraic group defined over $\mathbb{R}$. Then there is a finite algorithm to compute $\widehat{G}_u$.*

There is an enormous difference between the conclusion of Theorem 1.3 and a computer program to implement it. In fact implementing such an algorithm will certainly require some new mathematical ideas. Much of the computational portion of this proposal can be summed up in

Problem 1.4. Write a computer program to compute the unitary dual for the group of real points of any connected reductive algebraic group defined over $\mathbb{R}$.

2. History of the problem

In order to introduce the history of Problem 1.2, as well as to set the stage for the mathematical questions to be addressed in the proposal, we begin with a little more detail about Gelfand's program for using symmetry to do mathematics. We take for the setting a locally compact group G acting (as a symmetry group) on a measure space X, preserving the measure. In this setting there is a Hilbert space $\mathcal{H} = L^2(X)$ of (complex-valued) square-integrable functions on X. Each element $g \in G$ defines a linear transformation $\pi(g)$ of $\mathcal{H}$, by the rule

$$[\pi(g)f](x) = f(g^{-1} \cdot x).$$

These linear transformations have three fundamental properties:

$$\pi(gh) = \pi(g)\pi(h), \qquad \pi(e) = I \qquad (g, h \in G); \tag{2.1)(a}$$

$$\text{for every } g \in G,\ \pi(g) \text{ is a unitary operator on } \mathcal{H}; \tag{2.1)(b}$$

$$G \times \mathcal{H} \to \mathcal{H}, \qquad (g, v) \mapsto \pi(g)v \quad \text{is continuous.} \tag{2.1)(c}$$

(The last condition requires some mild assumptions about the "niceness" of the action of G on the measure space X.)

The central idea of Gelfand's program is to replace the (non-linear) problem of studying group actions on sets by the (linear) problem of studying families of linear transformations satisfying these three properties. The first step in the program is to express questions about X (related to the symmetry group G) as questions about $L^2(X)$ (and the linear operators $\pi(g)$).

A *unitary representation of* G is a pair $(\pi, \mathcal{H})$ consisting of a complex Hilbert space $\mathcal{H}$ and a mapping π from G to linear transformations of $\mathcal{H}$, subject to the three conditions (2.1). An *invariant subspace* of π is a closed subspace $\mathcal{V} \subset \mathcal{H}$ that is preserved by all of the linear transformations $\pi(g)$. In this case the the orthogonal complement $\mathcal{V}^\perp$ is also invariant, and

$$\pi = \pi_{\mathcal{V}} \oplus \pi_{\mathcal{V}^\perp}$$

in an obvious sense. Such a decomposition puts the operators $\pi(g)$ simultaneously in block-diagonal form, and so reduces many questions about π to questions about $\pi_{\mathcal{V}}$ and $\pi_{\mathcal{V}^\perp}$ separately. A unitary representation $(\pi, \mathcal{H})$ is called *irreducible* if it has precisely two invariant subspaces (which must then be $\mathcal{H}$ and 0). The second step in Gelfand's program is to decompose an arbitrary unitary representation of G as (something like) a direct sum of irreducible unitary representations; and to express the original questions about X as questions about each of the irreducible representations appearing in the decomposition. This step is a general version of "harmonic analysis." The existence of such a decomposition (as a "direct integral" rather than a direct sum) was established by the 1950s in very great generality using the theory of rings of operators (see for example [Dix]). Making the decomposition explicit in the case of interesting groups acting on interesting measure spaces is a large and ongoing research area; we will not address it here. (There is a long history of important influences of harmonic analysis on the classification of irreducible unitary representations. Harish-Chandra constructed the "discrete series" of irreducible unitary representations of a reductive Lie group G in the course of his explicit decomposition of $L^2(G)$. Arthur's conjectures about "unipotent representations" for reductive groups over local fields arise from his analysis of Langlands' partially explicit decomposition of $L^2(\boldsymbol{G}(\mathbb{A})/\boldsymbol{G}(\mathbb{Q}))$.)

The third step in Gelfand's program is description of all irreducible unitary representations of G. Recall that in step two, questions about X were broken into questions about each of the irreducible unitary representations appearing in $L^2(X)$. What "description" means should therefore be that we can answer these questions; that is, there will be a different meaning depending on what questions we were originally asking about X. The tools we want to build to describe irreducible unitary representations should be able to answer some interesting questions about them; we will give examples of such questions as we proceed.

The fourth and final step in Gelfand's program is to assemble the answers to questions about irreducible representations into answers to our original questions about X. Several examples of how this program works are described in [VR]

Modulo small algebraic difficulties, one can reduce Problem 1.2 to the case of simple groups:

Problem 2.2. For every almost simple algebraic group $\boldsymbol{G}$ defined over a local field F, determine the set $\widehat{\boldsymbol{G}(F)}_u$ of irreducible unitary representations $\boldsymbol{G}(F)$.

Here is a brief summary of what is known about Problem 2.2. For $GL_n(F)$, it is solved completely: in [Ta] for p-adic fields, and in [VG] for the real and complex fields. (To be precisely in the context of Problem 2.2 we should talk about $SL_n(F)$. For those groups the answer is not quite so perfectly understood, because of the algebraic problems attached to restricting representations from $GL_n(F)$ to $SL_n(F)$.) For other classical groups (types B, C, and D) and the complex field, it is solved in [BC]. The problem has been solved for a relatively small number of additional groups, and there are various kinds of partial results especially in the case of the real and complex fields; some of these will be described below.

Even partial results can be powerful tools in Gelfand's program. For example, it often happens that the answer to some question about X is expressed as a sum over contributions from various irreducible representations appearing in $L^2(X)$; and that the contribution of an irreducible representation is zero unless the representation has some unusual property. Then one can hope to analyze just the irreducible unitary representations having that unusual property, and this is often easier than the general Problem 2.2. One example is the analysis of the cohomology of locally symmetric spaces given in [VZ]. A more recent example is Jian-shu Li's work [Li] on the first eigenvalue of the Laplacian on locally symmetric spaces. This latter example is particularly relevant to this proposal, because the unitary representations that must be analyzed in Li's case (the "spherical" ones) are exactly those most susceptible to the methods we will describe.

3. Technical background

Suppose now that G is the group of F-points of a connected reductive algebraic group defined over a local field F. Let K be a maximal compact subgroup of G (if F is $\mathbb{R}$ or $\mathbb{C}$), or an open compact subgroup (if F is non-archimedean). In this section we will describe results that make the study of irreducible unitary representations of G accessible to finite calculations.

Any unitary representation $(\pi, \mathcal{H})$ of G may be restricted to a unitary representation of K. Now any irreducible unitary representation of K is finite-dimensional, and any unitary representation of K is a Hilbert space direct sum of irreducible representations. We can therefore write

$$\mathcal{H} = \oplus_{\mu \in \widehat{K}_u} \mathcal{H}(\mu) \qquad \text{(Hilbert space direct sum)}, \tag{3.1)(a}$$

with $\mathcal{H}(\mu)$ a sum of copies of the irreducible representation μ. One can therefore attach to $(\pi, \mathcal{H})$ in a canonical way the pre-Hilbert space

$$X(\pi) = \sum_{\mu \in \widehat{K}_u} \mathcal{H}(\mu) \qquad \text{(algebraic direct sum)}. \tag{3.1)(b}$$

Harish-Chandra showed how π provides some additional algebraic structure on $X(\pi)$. He axiomatized this additional structure as a purely algebraic theory of *admissible representations* of G. We refer to [BW] (pages 6 and 290) for the definitions. In particular,

$$\text{If } (\pi, \mathcal{H}) \in \widehat{G}_u, \text{ then } X(\pi) \text{ is an irreducible admissible representation of } G. \tag{3.2}$$

Theorem 3.3 (Harish-Chandra). *Suppose G is the set of F-points of a connected reductive algebraic group defined over a local field F. There is a natural bijection between the set $\widehat{G}_u$ of irreducible unitary representations of G, and the set of irreducible admissible representations of G admitting a positive definite invariant Hermitian form*

Because of the bijection in Theorem 3.3, it is now reasonable to define

$$\widehat{G} := \text{irreducible admissible representations of } G \tag{3.4)(a}$$

$$\widehat{G}_h := \text{irreducible admissible Hermitian representations of } G \tag{3.4)(b}$$

$$\widehat{G}_u := \text{irreducible admissible unitary representations of } G, \tag{3.4)(c}$$

in each case up to equivalence of admissible representations. The definitions provide obvious inclusions

$$\widehat{G}_u \subset \widehat{G}_h \subset \widehat{G}. \tag{3.4)(d}$$

By the early 1970s, Langlands in [La] gave a fairly explicit classification of $\widehat{G}$ (which we will recall in a moment) in the case of archimedean F. His ideas provided a somewhat less explicit classification for non-archimedean F. Knapp and Zuckerman first understood that the Langlands classification almost automatically provides at the same time a classification of the Hermitian representations $\widehat{G}_h$. The problem that remains is to look at each Hermitian representation, and to decide whether the invariant Hermitian form on it is definite. This is the strategy for classifying unitary representations presented in [KZ], and it has been the framework for most subsequent work on the problem. In order to explain what we propose to do, we need to state the Langlands classification with some care. More details may be found in [BW].

Definition 3.5. A *rational Levi subgroup* L of G is the centralizer in G of an F-split torus. (This terminology is not standard and not perfect. The hypothesis means that L is the group of F-points of a Levi factor of a parabolic subgroup defined over F, which is much stronger than that L is the group of F-points of a Levi subgroup defined over F.) Write A for the largest F-split torus in the center of L. This is a product of copies of $F^\times$, so its lattice of rational one-parameter subgroups is a lattice $\mathfrak{a}_{\mathbb{Z}} = \mathfrak{a}_{\mathbb{Z}}(L)$ of finite rank. The real vector space

$$\mathfrak{a}_0^* = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}_{\mathbb{Z}}, \mathbb{R})$$

can be naturally identified with the one-dimensional characters of L taking positive real values; if $\nu \in \mathfrak{a}_0^*$, then we write

$$e^\nu \colon L \to \mathbb{R}^{+,\times}$$

for the corresponding character.

An irreducible admissible representation δ of L is said to be in the *relative discrete series* if it is unitary, and the matrix coefficients (of the corresponding irreducible unitary representation of L) are square-integrable modulo A.

Suppose δ is a relative discrete series for L and $\nu \in \mathfrak{a}_0^*$. Choose a parabolic subgroup $P = LN$ of G so that ν is weakly positive on the roots of A in N (see [BW] for details). The *standard representation (of quotient type) with parameters δ and ν* is

$$X_q(\delta, \nu) := \mathrm{Ind}_P^G(\delta \otimes e^\nu \otimes 1).$$

(The ambiguity in the choice of P does not affect this representation.) This is an admissible representation of G having a finite composition series. The *Langlands quotient* is defined to be the largest completely reducible quotient representation $\overline{X}(\delta, \nu)$ of $X_q(\delta, \nu)$.

Write $P^{op} = LN^{op}$ for the opposite parabolic subgroup to P. The *standard representation (of sub type) with parameters δ and ν* is

$$X_s(\delta, \nu) := \mathrm{Ind}_{P^{op}}^G(\delta \otimes e^\nu \otimes 1).$$

This is an admissible representation of G having a finite composition series; the composition factors and multiplicities are the same as for $X_q(\delta, \nu)$. The *Langlands subrepresentation* is defined to be the largest completely reducible subrepresentation $\overline{X}(\delta, \nu)$. The notation is justified because there is a natural intertwining operator

$$A(\delta, \nu) \colon X_q(\delta, \nu) \to X_s(\delta, \nu)$$

(defined for now up to a scalar multiple) that carries the Langlands quotient on the left onto the Langlands subrepresentation on the right.

The proof of the next theorem is essentially due to Langlands, but this formulation reflects the work of several other people, including Miličić.

Theorem 3.6 (Langlands)**.** *Every irreducible admissible representation of G is a summand of some Langlands quotient representation $\overline{X}(\delta, \nu)$. The triple (L, δ, ν) is uniquely determined (by the original irreducible) up to conjugation in G.*

The Langlands classification provides a natural non-degenerate Hermitian pairing between $X_q(\delta, \nu)$ and $X_s(\delta, -\nu)$: each may be regarded as induced from the same parabolic P (which makes ν positive and $-\nu$ negative) and the pairing is an integration over the compact space G/P. It follows that there is a non-degenerate Hermitian pairing between $\overline{X}(\delta, \nu)$ and $\overline{X}(\delta, -\nu)$. Classifying Hermitian representations amounts (more or less) to understanding when these are equivalent, so we turn next to that question.

Definition 3.7. Suppose L is a rational Levi subgroup of G, with maximal F-split torus A. The *Weyl group of L* is

$$W(G, L) := N_G(L)/L = N_G(A)/Z_G(A).$$

It is sometimes convenient to write this group as $W(G, A)$. It is a finite group, but not a Coxeter group in general. Here is a way to study it. Extend A to a maximal F-split torus $A_{max} \subset L$. Then A_{max} is also a maximal F-split torus in G. The Weyl group $W(G, A_{max})$ is equal to the Weyl group of the "restricted roots," the system of roots of A_{max} in G. Then

$$W(G, A) = \{w \in W(G, A_{max}) | w(A) \subset A\}/W(L, A_{max}).$$

This allows one to compute $W(G, A) = W(G, L)$ by working in the Coxeter group $W(G, A_{max})$. (It is a simple matter to describe the possible subgroups $A \subset A_{max}$ in terms of the restricted root system.)

The finite group $W(G, L)$ acts on irreducible admissible representations of L, preserving the relative discrete series. It also acts on the real vector space $\mathfrak{a}_0^*$. Given a relative discrete series representation δ of L, write $W(G, L)_\delta$ for the stabilizer of δ in $W(G, L)$.

Theorem 3.8 (Knapp-Zuckerman; see [KZ])**.** *Suppose π is an irreducible admissible representation of G. Using Theorem 3.6, embed π in a Langlands quotient $\overline{X}(\delta, \nu)$, attached to the rational Levi subgroup L of G. Let $W(G, L)_\delta$ be the stabilizer of δ in the Weyl group of L (Definition 3.7). Then π admits an invariant Hermitian form only if there is an element $w \in W(G, L)_\delta$ such that $w\nu = -\nu$. If F is archimedean, this condition is also sufficient.*

In the setting of Theorem 3.8, Knapp and Zuckerman explain how to write down a non-degenerate Hermitian form on $\overline{X}(\delta, \nu)$ using the Weyl group element w and the intertwining operator of Definition 3.5. The difficulty in the non-archimedean case is that this form may restrict to zero on the subrepresentation π; that is why we cannot say that the condition of the theorem is sufficient for π to be Hermitian.

To pass from Theorem 3.8 to an explicit description of the unitary dual of G, there are a host of computational issues. We have indicated that describing the (finitely many) possible groups A and L is not difficult. We need then an explicit parametrization of the relative discrete series of L. For archimedean fields this is provided by the work of Harish-Chandra. For non-archimedean fields the situation is far more complicated. There are good parametrizations of the relative discrete series only for $GL(n)$, some closely related groups, and a few small examples. This problem is not one that we intend to address, but we can indicate briefly why it is not a reason to abandon all hope. For this it is convenient to use a picture of $\widehat{G}$ that differs slightly from the Langlands classification, and that is available *only* in the non-archimedean case. The Langlands classification realizes any representation as a special kind of subquotient of something induced from a discrete series representation. One can instead realize any representation as a general subquotient of something induced from a supercuspidal representation. The basic data are pairs (L, σ), with L a rational Levi subgroup and σ a unitary relative supercuspidal representation of L. The main theorem says that every irreducible admissible representation appears as a subquotient of some

$$\mathrm{Ind}_P^G(\sigma \otimes e^\nu \otimes 1),$$

with $\nu \in \mathfrak{a}_0^*$, and that the triple (L, σ, ν) is uniquely determined up to conjugation in G. (The parameters are a proper subset of those in Theorem 3.6; the classification still works because we allow all possible subquotients of the induced representations.) With this parametrization, each pair (L, σ) defines a connected component of $\widehat{G}$; twisting σ by an unramified unitary character of L does not change the connected component. The Bushnell-Kutzko theory of types ([BK]) seeks to attach to such a component a particularly nice open compact subgroup K_1 and irreducible representation μ_1 of K_1. Questions about this connected component of $\widehat{G}$ (like which representations are Hermitian or unitary) can be translated into questions about representations of the Hecke algebra defined by G, K_1, and μ_1. One can then hope that this Hecke algebra is isomorphic to one attached in a similar fashion to a smaller group G', and a pair (L', σ'); the group A' will presumably be naturally isomorphic to A. (There are results of this kind due to Kim [KM] and many others.) In the best of all possible worlds, L' is a minimal rational Levi factor in G', so it is compact modulo its center. This is now getting close to the setting in which we *will* be able to say something about unitary representations (see section 4).

To say more about computational issues, a little notation is helpful. Suppose X is an admissible representation of G, and (μ, E_μ) is an irreducible unitary representation of K. Define

$$X^\mu = \mathrm{Hom}_K(E_\mu, X), \tag{3.9)(a}$$

a vector space of dimension equal to the multiplicity $m_X(\mu)$ of μ in X. If X is Hermitian, then X^μ inherits a natural Hermitian form $\langle , \rangle^\mu$. This form has a signature given by three non-negative integers $p_X(\mu)$, $q_X(\mu)$, and $z_X(\mu)$, corresponding to the dimensions of a maximal positive definite subspace, a maximal negative definite subspace, and the radical. We have

$$p_X(\mu) + q_X(\mu) + z_X(\mu) = m_X(\mu), \tag{3.9)(b}$$

and the Hermitian form on X is positive semidefinite on the μ-isotypic part of X if and only if $q_X(\mu) = 0$.

Suppose now that we are in the setting of Definition 3.5, and assume for simplicity that K acts transitively on G/P. (This is automatic in the archimedean case, and such compact subgroups exist in the non-archimedean case.) It follows that

$$[\mathrm{Ind}_P^G(\delta \otimes e^\nu \otimes 1)]^\mu \simeq \mathrm{Hom}_{P\cap K}(E_\mu, \delta), \tag{3.9)(c}$$

a finite-dimensional space which is independent of ν. The Hermitian forms implicit in Theorem 3.8 provide a family of Hermitian forms $\langle , \rangle^\mu(\nu)$ on this vector space, with signatures $p_{\delta,\nu}(\mu), q_{\delta,\nu}(\mu), z_{\delta,\nu}(\mu)$. Determining when $\overline{X}(\delta, \nu)$ is unitary amounts to determining when all of the $q_{\delta,\nu}(\mu)$ (as μ varies) are equal to zero. The first computational issue is that there are infinitely many μ. In the p-adic case this issue is more or less subsumed in the discussion above: if an appropriate (μ_1, K_1) can be found, then one need only consider the finitely many μ whose restrictions to K_1 contain μ_1. (We will see an example at (4.14) below.)

Here is a theorem for the real case.

Theorem 3.10. *Suppose G is the group of real points of a connected reductive real algebraic group. In the setting of Definition 3.5, suppose X is an irreducible Langlands quotient of $X_q(\delta, \nu)$. Write λ for the Harish-Chandra parameter of δ, which we can regard as a purely imaginary linear functional on the Lie algebra of a maximal torus of K. Write ρ for the half sum of a set of positive roots making λ dominant.*

If X is not unitary, then there is a μ' in $\widehat{K}$ such that

1. *μ' is a lowest K-type of a standard representation $X_q(\delta', \nu')$;*
2. *the Langlands parameter λ' of δ' satisfies $|\lambda'| \le |\lambda + \rho|$; and*
3. *$q_X(\mu') > 0$.*

This theorem follows fairly easily from [VU] and [BW], Proposition V.2.2. Condition (2) describes a finite set of possible δ', and therefore a finite set of possible representations μ' of K. For a fixed series of representations (corresponding to a pair (L, δ)) it says that unitarity can be decided by computing a specified finite set of the integers $q_X(\mu')$.

The finite set provided by Theorem 3.10 is still much too large for our computational goals. For spherical representations of the complex group of type E_8, the largest μ' appearing in Theorem 3.10 is the representation of highest weight 2ρ, which has dimension 3^{120}. The multiplicity $m_X(\mu')$ of this representation of K in a generic spherical representation X is (according to the software package lie)

$$6508567580670893055947363315903329107522115068801$$

We do not anticipate being able to compute signatures of Hermitian matrices of this size. In section 5 we will give some indications of how we anticipate finding a much smaller set of μ' (with much smaller multiplicities) that still suffice for testing unitarity.

Once we know which representations of K we wish to consider, the next computational issue is writing down the Hermitian forms $\langle,\rangle^\mu(\nu)$ on $X_q(\delta,\nu)^\mu$. We will give some examples of dealing with this issue in sections 4 and 5. An important first step is simply writing down the vector space (cf. (3.9)(c)). In the real case, this step can be broken into two problems.

Problem 3.11. Suppose G is the group of real points of a connected reductive real algebraic group, L is a rational Levi subgroup stable under the Cartan involution, and (δ, X_δ) is a relative discrete series representation of L. For every irreducible representation (γ, F_γ) of $L\cap K$, describe the vector space

$$\mathrm{Hom}_{L\cap K}(F_\gamma, X_\delta).$$

Problem 3.12. Suppose G is the group of real points of a connected reductive real algebraic group, L is a rational Levi subgroup of G stable under the Cartan involution of G, and (μ, E_μ) is an irreducible representation of K. For every irreducible representation (γ, F_γ) of $L\cap K$, describe the vector space

$$\mathrm{Hom}_{L\cap K}(E_\mu, F_\gamma).$$

In both problems, we are being deliberately vague with the phrase "describe the vector space." At least what is required is an algorithm to calculate the dimension. What we want in the end is an explicit description of certain Hermitian forms on these vector spaces, and that requirement will control what constitutes a satisfactory solution of these two problems.

For Problem 3.11, at least the calculation of dimension is solved for connected L by the Blattner formula, proved by Hecht and Schmid. For disconnected L, the main point is to get an appropriate parametrization of $\widehat{L\cap K}$, and to understand how to restrict representations of $L\cap K$ to $(L\cap K)_0$. These are problems that we understand fairly well. However the Blattner formula involves a sum over the Weyl group and a partition function; as such it becomes unwieldy, for example for E_8.

Problem 3.12 is surprisingly poorly understood. For complex groups it asks for branching laws for reductive Lie algebras sharing a common Cartan, and existing software such as lie can compute these dimensions. For $GL(n,\mathbb{C})$, for example, what is needed is branching laws from $U(p+q)$ to $U(p)\times U(q)$. The multiplicities are given by the Littlewood-Richardson rule, and are well understood. For real groups the situation is much worse. Even for $GL(n,\mathbb{R})$, Problem 3.12 amounts to having explicit branching laws from $O(p+q)$ to $O(p)\times O(q)$ (with $p+q=n$). Such branching laws are in some sense implicit in Weyl's description of the representations of $O(n)$ in [W]; explicit versions appear in the work of R. C. King [K], but to our knowledge they have not been implemented in publically available software. Ideas like King's are probably sufficient to solve Problem 3.12 for classical G. For exceptional real groups, the disconnectedness of $L\cap K$ means that at least a little new mathematics will be required even to write algorithms to solve the problem.

We have now given a very rough outline of a finite computational procedure for deciding whether a particular Langlands subquotient of a standard representation $X(\delta,\nu)$ is unitary in the archimedean case. Since there are infinitely many parameters δ and ν, this is a long way from a finite procedure for determining the unitary dual (Theorem 1.3). Here is a sketch of what is missing.

First we fix δ and consider the possibilities for ν. For a fixed δ, the various Hermitian representations $X_q(\delta,\nu)$ are realized on a common vector space, with Hermitian form defined using an intertwining operator that depends nicely on ν. Zeros of the intertwining operator occur along certain rational hyperplanes in the parameter ν, which can be determined explicitly. The hyperplanes partition the possible ν into a finite number of explicitly described cells, in such a way that signatures of Hermitian forms are constant on each cell. We need only determine which cells correspond to unitary representations, and for this it is enough to consider a single point ν in each of the cells. See the discussion following (4.14) for more details in the case of spherical representations.

Problem 3.13. Understand the cell decompositions coming from studying reducibility of intertwining operators.

This problem has both theoretical and computational aspects.

We have now seen that the unitary representations corresponding to a fixed (L, δ) may be found by a finite computation. To deal simultaneously with families of δ, we need a slightly different argument. We are still only in the archimedean case. It turns out that one can attach to the pair (L, δ) various subgroups H of G (the centralizers of certain compact tori inside L). Then $L \cap H$ is a rational Levi subgroup of H. The Harish-Chandra parameter for δ provides also a relative discrete series representation δ_H of $L \cap H$. The groups A for (G, L) and $(H, L \cap H)$ are the same, and we can arrange for the stabilizer of δ in $W(G, A)$ to be contained in the subgroup $W(H, A)$:

$$W(G, A)_\delta = W(H, A)_{\delta_H}. \tag{3.14)(a}$$

Then $\overline{X}_G(\delta, \nu)$ is obtained in a well understood way (Zuckerman's derived functors) from $X_H(\delta_H, \nu)$. [**Note to PI's: I think little bit about derived functors here would be useful (jda)**] So far this is rather easy, and can be arranged with many choices of H. With more care, it is possible to choose H in such a way that

$$\overline{X}_G(\delta, \nu) \text{ is unitary if and only if } \overline{X}_H(\delta_H, \nu) \text{ is unitary.} \tag{3.14)(b}$$

Of course one way to achieve all of this is to take $H = G$. What follows from [VU] is essentially

$$\begin{array}{l}\text{for all but finitely many pairs } (L, \delta)\text{, we can choose} \\ H \text{ of strictly smaller dimension than } G.\end{array} \tag{3.14)(c}$$

One way to summarize this is:

Theorem 3.15. *There is a finite set of K-types S with the following property. Suppose X is an irreducible representation with real infinitesimal character. If a minimal K–type of X is not contained in S then there is a group H of strictly smaller dimension, and a representation X_H of H so that X is unitary if and only if X_H is unitary.*

According to [SV] we may conjecturally take S to be the set of K–types whose highest weights are contained in the convex hull of $W\rho$. In the absence of this conjecture the set S may be much larger, and computationally infeasible.

Therefore the computation of the unitary dual of G reduces to the computation of the unitary representations corresponding to a finite set of (L, δ) for G and for the subgroups H of G.

For example if X is a spherical representation then $H = G$ and there is no reduction. This case will be discussed in more detail in Section 4. By the preceding discussion the full unitary dual will follow from a finite number of calculations of this type. See Section 5 for more information about the general case.

This completes our sketch of the proof of Theorem 1.3.

The general theme of identifying parts of the unitary dual of G with parts of the unitary duals of smaller groups H appeared already in our discussion of the Bushnell-Kutzko theory of types, and it plays a large role in the Barbasch-Moy theory discussed in section 4. Results of the form (3.14)(b) are in some respects even more satisfactory than lists of unitary representations, because they have explanatory content.

4. The spherical unitary dual for split groups: p-adic case

In this section we discuss the spherical unitary representations of split groups over p–adic and real fields. We emphasize this case for two reasons. On the one hand it is in some sense the simplest; for example the parametrization of spherical representations of a split group $\boldsymbol{G}(F)$ spherical is essentially independent of the field F. On the other hand it is in some sense the hardest case serves as the prototype for much more general results. For example in the classification of the unitary dual of $GL(n)$, the spherical case is the most important one.

Let F be $\mathbb{R}$ or a p–adic field, and let $G = \boldsymbol{G}(F)$ be the F–points of a split reductive algebraic group defined over F. Let K be a maximal compact subgroup of G (see (4.2)(a) for precise assumptions in the p–adic case). We say a representation π of G is *spherical* if it contains a K–fixed vector.

Problem 4.1. Classify the irreducible spherical unitary representations of G.

The answer is known for all classical groups over a real or p–adic field by recent work of Dan Barbasch [B3], and for G_2, F_4, and E_6 over the p–adics by work of Barbasch and Dan Ciuboataru. This has also been confirmed by computer calculation; see the end of Section 4A. In all known cases the answers are "the same", providing a strong example of the Lefschetz princple.

4A p–adic case. Now let F be a p–adic field. Write

$$F \supset \mathcal{R} \supset \mathcal{P} = \varpi\mathcal{R},$$

where $\mathcal{R}$ is the ring of integers in F and $\mathcal{P}$ its unique maximal ideal. Then $\mathcal{R}/\mathcal{P}$ is the finite field $\mathbb{F}_q$ with q elements. The group $\boldsymbol{G}$ may be regarded as defined over $\mathcal{R}$. Choosing such a structure provides a distinguished maximal compact subgroup

$$K := \boldsymbol{G}(\mathcal{R}), \tag{4.2)(a}$$

which contains a normal subgroup

$$K_1 := \{g \in K \mid g \equiv e(mod\ \mathcal{P})\}. \tag{4.2)(b}$$

It turns out that

$$K/K_1 \simeq \boldsymbol{G}(\mathbb{F}_q), \tag{4.2)(c}$$

a split finite Chevalley group.

An admissible irreducible representation (π, V) is called *spherical* if $V^K \neq (0)$. One reason that such representations are of particular importance is that automorphic representations for split groups are necessarily spherical at all but finitely many places.

The set of spherical admissible representations is not closed in the admissible dual of G. Just as in the discussion after Theorem 3.8, it is convenient to consider the slightly larger class called *Iwahori spherical* representations, which *is* closed. An *Iwahori subgroup* of G is by definition any open compact subgroup which is conjugate to the inverse image in K of a Borel subgroup of $G(\mathbb{F}_q)$ (cf. (4.2)(c)). Fix an Iwahori subgroup $\mathbb{I}$. Let $\mathcal{C}(\mathbb{I})$ be the category of admissible representations of G with the property that each subquotient is generated by its $\mathbb{I}$-fixed vectors. Let

$$\mathcal{H}(G//\mathbb{I}) := \{f \in C_c(G) |\ f(i_1 g i_2) = f(g) \quad (i_1, i_2 \in \mathbb{I})\}. \tag{4.3}$$

Then $\mathcal{H}(G//\mathbb{I})$ is an algebra under convolution, and $V^{\mathbb{I}}$ is a representation of this algebra, for any admissible representation V of G.

Theorem 4.4 (Borel-Casselman, [Bo])**.** *Suppose G is a split reductive p-adic group, and $\mathbb{I}$ is an Iwahori subgroup. The functor*

$$V \longrightarrow V^{\mathbb{I}}$$

is an equivalence of categories between $\mathcal{C}(\mathbb{I})$ and the category of finite-dimensional representations of $\mathcal{H}(G//\mathbb{I})$. In particular, irreducible representations of G with a non-zero $\mathbb{I}$-fixed vector are in one-to-one correspondence with simple modules for the Iwahori Hecke algebra $\mathcal{H}(G//\mathbb{I})$.

If V is unitary, then so is $V^{\mathbb{I}}$. The converse is less trivial.

Theorem 4.5 ([BM1])**.** *The Iwahori-spherical admissible representation irreducible representation V is unitary if and only if $V^{\mathbb{I}}$ is unitary as well.*

This result reduces the problem of unitarity for the infinite-dimensional representations V to the problem of unitarity for the finite-dimensional modules for $\mathcal{H}(G//\mathbb{I})$.

The structure of $\mathcal{H}(G//\mathbb{I})$ is well understood in terms of generators and relations. We do not give it here. Instead we describe the affine graded Hecke algebra $\mathbb{H}$; by [BM2], the problem of determining the unitary dual of $\mathcal{H}(G//\mathbb{I})$ can be reduced to the corresponding problem for $\mathbb{H}$ and for similar algebras attached to various smaller split groups.

Because G is split, a Levi component of a minimal parabolic subgroup of G is a split torus A. We use notation as in Definition 3.5. Let R be the set of roots of A in G, R^+ a set of positive roots, and Π the corresponding set of simple roots. Let $W = W(G, A) = W(R)$ be the Weyl group. The *affine graded Hecke algebra* as a vector space is

$$\mathbb{H} = \mathbb{C}[W] \otimes S(\mathfrak{a}), \tag{4.6)(a}$$

with $\mathfrak{a} = \mathfrak{a}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ the complexification of the lattice of Definition 3.5. The algebra structures on $\mathbb{C}[W]$ and the symmetric algebra $S(\mathfrak{a})$ are the usual ones. In addition, if we denote by $s_\alpha \in W$ the reflection corresponding to $\alpha \in \Pi$ and t_α the corresponding element in $\mathbb{C}[W]$, then

$$\omega t_\alpha - t_\alpha s_\alpha(\omega) = (\omega, \check{\alpha}), \quad \omega \in \mathfrak{a}. \tag{4.6)(b}$$

There is a $*$ operation on $\mathbb{H}$ given by

$$w^* = w^{-1}, \qquad \omega^* = -\omega + \sum_{\beta \in R^+} t_\beta(\omega, \check{\beta}), \qquad \omega \in \mathfrak{a}_0. \tag{4.6)(c}$$

Write $\mathbb{A}$ for $S(\mathfrak{a})$.

A module for $\mathbb{H}$ is at the same time a module for $\mathbb{C}[W]$—that is, a representation of W—and a module for the symmetric algebra $\mathbb{A}$. These two actions are related by the commutation relations (4.6). In the correspondence established in [L3] and [BM2] between Iwahori-spherical representations and $\mathbb{H}$-modules, spherical representations of G correspond to $\mathbb{H}$-modules containing a non-zero W-fixed vector.

The one-dimensional modules for $\mathbb{A}$ correspond to elements $\nu \in \mathfrak{a}^*$. It follows easily that any $\mathbb{H}$-module must appear as a subquotient of some induced module

$$X(\nu) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\nu, \tag{4.7)(a}$$

which we call a *principal series module.* There is a canonical isomorphism of W-modules

$$X(\nu) \simeq \mathbb{C}[W]. \tag{4.7)(b}$$

In particular, $X(\nu)$ has a unique W-fixed vector, and therefore a unique irreducible spherical subquotient $\overline{X}(\nu)$.

Here is a version of Theorem 3.6 for $\mathbb{H}$-modules.

Theorem 4.8. *Any irreducible $\mathbb{H}$-module is a subquotient of a principal series $X(\nu)$.*

1. *If $\operatorname{Re}\nu$ is dominant with respect to R^+, then $X(\nu)$ has a unique irreducible quotient, and this quotient is equal to $\overline{X}(\nu)$.*
2. *If $\operatorname{Re}\nu$ is antidominant, then $X(\nu)$ has a unique irreducible submodule, and this submodule is equal to $\overline{X}(\nu)$.*
3. *The modules $\overline{X}(\nu)$ and $\overline{X}(\nu')$ are equivalent if and only if there is $w \in W$ such that $w\nu = \nu'$. If that is the case, then the full principal series modules $X(\nu)$ and $X(\nu')$ have exactly the same composition factors and multiplicities.*

For the study of spherical unitary modules, the reduction arguments from [L3] and [BM2] allow us to consider only $\nu \in \mathfrak{a}_0^*$, and we assume this from now on. There is an analogue of the intertwining operator $A(\delta, \nu)$ (introduced before Theorem 3.6), constructed as follows. Let

$$r_\alpha := (t_\alpha \alpha - 1)(\alpha - 1)^{-1}. \tag{4.9)(a}$$

This is an element in the algebra analogous to $\mathbb{H}$ where $\mathbb{A}$ is replaced by its field of fractions $\mathcal{F}$. The r_α (for α simple) satisfy the braid relations as well as

$$\omega r_\alpha = r_\alpha s_\alpha(\omega) \qquad (\omega \in \mathfrak{a}). \tag{4.9)(b}$$

Thus if $w \in W$ and $w = s_1 \dots s_k$ is a reduced decomposition, then $r_w = r_1 \dots r_k$ is well-defined. Write $r_w(\nu) \in \mathbb{C}[W]$ for the expression where r_w has been expanded into a sum $\sum t_w f_w$ with $f_w \in \mathcal{F}$, and the f_w have been evaluated at ν. Then

$$A_w(\nu)\colon X(\nu) \longrightarrow X(w\nu), \qquad x \otimes \mathbb{1}_\nu \mapsto x r_w(w\nu) \otimes \mathbb{1}_{w\nu} \tag{4.9)(c}$$

is an intertwining operator. Since $r_w(w\nu)$ has poles, $A_w(\nu)$ is not defined for all values of ν. The poles turn out to lie on hyperplanes $\langle \nu, \check{\alpha} \rangle = -1$, with α a positive root. In particular, $A_w(\nu)$ is defined whenever $\nu \in \mathfrak{a}_0^*$ is dominant. Assume this is the case, and let w_0 be the long element of W. Then $X_q := X(\nu)$ and $X_s = X(w_0\nu)$ are analogous to the standard representations of quotient and submodule type, and

$$A_{w_0}(\nu)\colon X_q \to X_s \tag{4.9)(d}$$

is analogous to the operator $A(\delta, \nu)$.

Theorem 3.8 takes the following sharper form. Let

$$\epsilon\colon \mathbb{C}[W] \longrightarrow \mathbb{C}, \qquad \epsilon(\sum a_w t_w) = a_1, \tag{4.10}$$

be the augmentation map.

Theorem 4.11. *Suppose $\nu \in \mathfrak{a}_0^*$ is dominant. Then the spherical representation $\overline{X}(\nu)$ is Hermitian if and only $w_0\nu = -\overline{\nu}$. In this case, the Hermitian form (pulled back to the principal series $X(\nu)$ of which $\overline{X}(\nu)$ is a quotient) is given by*

$$\langle x \otimes \mathbb{1}_\nu, y \otimes \mathbb{1}_\nu \rangle = \epsilon(y^* r_{w_0}(w_0\nu)^* x).$$

A more explicit form is given by the following. Because of $(4.7)(b)$,

$$X(\nu) = \sum_{\tau \in \widehat{W}} V_\tau \otimes V_\tau^*, \tag{4.12)(a}$$

with the action of W on the first factor only. Fix a positive definite W-invariant form on each V_τ. Then the intertwining operator $r_{w_0}(\nu)$ induces a Hermitian operator

$$a_{w_0}(\tau, \nu)\colon V_\tau^* \longrightarrow V_\tau^* \tag{4.12)(b}$$

on the Weyl group representation V_τ^*. If we are given a model for the representation τ^* on $\mathbb{C}^n$ (with explicit matrices for the generating simple reflections in W), then the formulas in (4.9) make it possible to compute the $n \times n$ matrix $a_{w_0}(\tau, \nu)$ as a rational function of ν. A little more explicitly, $a_{w_0}(\tau, \nu)$ is a product of operators like

$$a_{s_\alpha}(\tau, \nu) = \begin{cases} 1 & \text{on the 1 eigenspace of } \tau(t_\alpha), \\ \frac{1 - \langle \nu, \check{\alpha} \rangle}{1 + \langle \nu, \check{\alpha} \rangle} & \text{on the } -1 \text{ eigenspace of } \tau(t_\alpha). \end{cases} \tag{4.12)(c}$$

Problem 4.13. Suppose $R \supset R^+$ is a root system and a set of positive roots in a real vector space $\mathfrak{a}_0^*$. Write W for the Weyl group of R, and w_0 for the long element of W. For each dominant $\nu \in \mathfrak{a}_0^*$ such that $w_0\nu = -\nu$, and each irreducible representation τ, V_τ of W, let $a_{w_0}(\tau, \nu)$ be the Hermitian operator of $(4.12)(b)$. Determine for which ν all of these operators are positive semidefinite.

Theorem 4.11 implies that the elements ν in Problem 4.13 are exactly those for which the spherical $\mathbb{H}$-module $\overline{X}(\nu)$ is unitary. By Theorem 4.5 knowledge of these sets (also for certain smaller root systems, using [BM2]) gives a solution to Problem 4.1.

It is worthwhile to see how this description fits into the general framework described at (3.9). The spherical principal series for G is induced from a Borel subgroup $B = AN$. The representation δ can be any unramified unitary character of A, and our standard representation is

$$X(\delta, \nu) = \mathrm{Ind}_B^G(\delta \otimes e^\nu \otimes 1). \tag{4.14)a}$$

We can choose $\mathbb{I} = (B \cap K)K_1$ as our Iwahori subgroup. The condition that δ is unramified means that δ is trivial on $B \cap K$, so the K-types of $X(\delta, \nu)$ are exactly the representations of K containing the trivial representaion of $B \cap K$ (cf. (3.9)(c)). Theorem 4.5 allows us to study unitarity by considering only the finite set of representations of K containing the trivial representation of $\mathbb{I} = (B \cap K)K_1$. Because of (4.2), these may be identified with representations of $G(\mathbb{F}_q)$ containing the trivial representation of $B(\mathbb{F}_q)$. It is a classical fact that such representations are in one-to-one correspondence with representations of W:

$$\left\{\begin{matrix}\text{irreducible representations of } G(\mathbb{F}_q) \text{ contain-} \\ \text{ing the trivial representation of } B(\mathbb{F}_q)\end{matrix}\right\} \longleftrightarrow \widehat{W} \qquad (4.14)(b)$$

If $\mu(\tau) \in \widehat{G(\mathbb{F}_q)}$ corresponds to $\tau \in \widehat{W}$, then

$$X(\delta, \nu)^{\mu(\tau)} \simeq \mathrm{Hom}_{B(\mathbb{F}_q)}(E_{\mu(\tau)}, \mathbb{C}) \simeq V_\tau^*; \qquad (4.14)(c)$$

the first isomorphism is $(3.9)(c)$, and the second comes from the proof of $(4.14)(b)$. In case δ is trivial, the Hermitian form $\langle , \rangle^{\mu(\tau)}(\nu)$ on this space (described in general after (3.9)) is the one given by $a_{w_0}(\tau, \nu)$. (Non-trivial δ are dealt with in [BM2] by reduction to smaller groups, in a step we have skipped several times already.)

We can explain now why (for a fixed G) Problem 4.13 can be solved by a finite calculation. The matrices $a_{w_0}(\tau, \nu)$ can change signature only along hyperplanes $\langle \nu, \check{\alpha} \rangle = 1$ or 0, with α a positive root. These hyperplanes partition the -1 eigenspace of w_0 in the positive Weyl chamber into a finite number of cells, and the signature of $a_{w_0}(\tau, \nu)$ is constant on each cell. Each cell contains a rational point ν_j (meaning that all $\langle \nu_j, \check{\alpha} \rangle$ are rational numbers). If our model of τ is by matrices with rational entries (as can always be arranged), then the matrix $a_{w_0}(\tau, \nu_j)$ has rational entries. Its signature may therefore be computed by a finite process. Since there are only finitely many ν_j to consider, this is an algorithm to determine the unitary spherical dual of a split p-adic group; we call it the *direct method.*

The direct method has the advantage that it is easy to determine a set of sample points to test for unitarity. signatures change is contained in the intersection of the hyperplanes $\mathfrak{a}$. This arrangement of hyperplanes has many special features that one can try to exploit. The disadvantage is that one has to consider all representations of W (in E_8 the largest dimension is 7168). In addition it is difficult to interpret the answer in terms of the Kazhdan-Lusztig classification. J. Adams, J. Stembridge, and J-K. Yu have written computer programs that have carried out such calculations for some classical groups, G_2, F_4 and E_6. They are in the process of obtaining the answer for E_7, and obtaining some partial results for E_8.

Performing this computation has given us insight into the computational issues which arise. In particular it was necessary to construct models of all irreducible of Weyl groups, to compute the signature of a large symmetric integral matrix, and to parametrize and run efficiently over the cells discussed above. See the guide to the Atlas web site after the references.

As noted after Problem 4.1, it has been solved for classical groups and for G_2, F_4 and E_6, by somewhat different methods. Because of the form of the answer when does this way is quite different from the one provided by the direct method, it is not entirely elementary to compare them. This has been done, and the answers agree exactly.

Barbasch's solution for classical groups has been implemented in the *spherical unitary explorer.* See the guide to the Atlas web site.

4B. Real case.

Now we take $F = \mathbb{R}$. Since G is split it contains a Borel subgroup $B = AN$. We may assume that A is preserved by the Cartan involution. (This notation may be misleading for experts. For us the group A is a split real torus, or a product of copies of the multiplicative group of $\mathbb{R}$. It is therefore a product of its identity component A_0 (which is isomorphic to $\mathbb{R}^l$) and

$$A \cap K \simeq (\mathbb{Z}/2\mathbb{Z})^l.$$

Traditional real groups notation would write A for A_0 and M for $A \cap K$, so that the Borel subgroup would be MAN.) Spherical representations of G are precisely the Langlands subquotients of

$$X_B(\delta, \nu) = \mathrm{Ind}_B^G(\delta \otimes e^\nu \otimes 1), \tag{4.15)(a}$$

with δ a unitary character of A trivial on $A \cap K$ and $\nu \in \mathfrak{a}_0^*$. Just as for spherical representations in the p-adic case, the study of unitarity can be reduced to the case when δ is trivial **[Is this really true? (jda)]** , so we assume that from now on . We are therefore studying the standard representations of quotient type

$$X_q(\nu) = \mathrm{Ind}_B^G(triv \otimes e^\nu \otimes 1), \qquad (\nu \in \mathfrak{a}_0^* \text{ dominant}) \tag{4.15)(b}$$

The corresponding standard representations of submodule type are

$$X_s(\nu) = \mathrm{Ind}_{B^{op}}^G(triv \otimes e^\nu \otimes 1), \qquad (\nu \in \mathfrak{a}_0^* \text{ dominant}) \tag{4.15)(c}$$

The Langlands subquotient $\overline{X}(\nu)$ is irreducible; it is the unique irreducible subquotient having a non-zero K-fixed vector. Just as in the p-adic case, $\overline{X}(\nu)$ is Hermitian if and only if $w_0\nu = -\nu$. (This is not just a formal similarity: if the real and p-adic groups have the same root data, then the parameter sets are exactly the same.)

The study of Hermitian forms requires the intertwining operator from Definition 3.5. Here is an explicit description. The integral operator

$$A(\nu)\colon X_q(\nu) \longrightarrow X_s(\nu), \qquad A(\nu)f(g) = n(\nu)\int_{N^{op}} f(gn)\, dn \tag{4.16)(a}$$

is well defined for ν dominant and regular. We choose the normalizing function $n(\nu)$ so that $A(\nu)$ is the identity on the K-fixed vector. (More precisely, there is a function f_q in X_q which is equal to 1 on K, and a function f_s in X_s which is 1 on K. We require $A(\nu)f_q = f_s$.) The operator $A(\nu)$ has a meromorphic continuation which is analytic for all ν dominant. This is the operator of Definition 3.5.

If (μ, E_μ) is any irreducible representation of K, then

$$X_q(\nu)^\mu = \mathrm{Hom}_K(E_\mu, X_q(\nu) \simeq \mathrm{Hom}_{A\cap K}(E_\mu, \mathbb{C}) \simeq X_s(\nu)^\mu \tag{4.16)(b}$$

(cf. (3.9)(c)). The intertwining operator $A(\nu)$ therefore induces an operator

$$a(\mu, \nu) \in \mathrm{End}(\mathrm{Hom}_{A\cap K}(E_\mu, \mathbb{C})), \tag{4.16)(c}$$

which is Hermitian whenever $w_0\nu = -\nu$.

Now the space in (4.16)(b) is

$$\mathrm{Hom}_{A\cap K}(E_\mu, \mathbb{C}) \simeq (E_\mu^{A\cap K})^*, \tag{4.16)(d}$$

Because every element of the Weyl group has a representative in K, it follows that

$$W \simeq N_K(A)/Z_K(A) = N_K(A)/(A \cap K)$$

must act on $E_\mu^{A\cap K}$. Write $\tau(\mu)$ for the representation of W on this space. In this way we can associate to every representation μ of K a (possibly reducible) representation $\tau(\mu)$ of W, with the property that

$$\dim \tau(\mu) = \text{multiplicity of } \mu \text{ in } X_q(\nu). \tag{4.16)(e}$$

The general machinery described in (3.9) says that

> for $\nu \in \mathfrak{a}_0^*$ dominant, the spherical representation $\overline{X}(\nu)$ is unitary if and only if $w_0\nu = -\nu$, and each of the Hermitian operators $a(\mu, \nu)$ (as μ varies) is positive semidefinite. (4.16)(f)

Theorem 3.10 allows us to restrict attention to a finite set of μ. We will show in this section how to compute the operators $a(\mu,\nu)$; how the nature of this computation points to a finite collection of special K-types (smaller than the set in Theorem 3.10); how the computation on those special K-types is related to Problem 4.13; and what conclusions we can draw about the spherical unitary dual.

The operator $A(\nu)$ has a factorization corresponding to a reduced decomposition of w_0. We will not recall that factorization, but instead will pass directly to writing down the corresponding factors of the operators $a(\mu,\nu)$ of (4.16)(c). Let $\alpha \in \Pi$ be a simple root of A, and $L_\alpha \supset A$ the corresponding Levi subgroup. There is a homomorphism

$$\phi_\alpha \colon SL(2,\mathbb{R}) \to L_\alpha, \tag{4.17)(a}$$

and L_α is locally isomorphic to the product of $\operatorname{im}\phi_\alpha$ and the central torus A_α. A little more precisely,

$$L_\alpha \cap K = \phi_\alpha(SO(2)) \cdot (A \cap K), \tag{4.17)(b}$$

with the first factor a normal subgroup. The intersection of the two factors is

$$\phi_\alpha(SO(2)) \cap A = \phi_\alpha(\pm I) = \{e, m_\alpha\}. \tag{4.17)(c}$$

An element $m \in A \cap K$ acts on $\phi_\alpha(SO(2))$ by

$$mxm^{-1} = x^{\alpha(m)} = x^{\pm 1} \qquad (x \in \phi_\alpha(SO(2))). \tag{4.17)(d}$$

Now the irreducible representations of $SO(2)$ are naturally parametrized by integers:

$$\chi_j(x) = e^{ij\theta}, \qquad x = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.18)(a}$$

Any representation (μ, E_μ) of K has a corresponding orthogonal decomposition

$$E_\mu = \sum_{j\in\mathbb{Z}} E_\mu(j), \qquad E_\mu(j) = \{v \in E_\mu | \mu(\phi_\alpha(x))v = \chi_j(x)v\} \tag{4.18)(b}$$

The commutation relations (4.17)(d) show that $A \cap K$ preserves $E_\mu(j) + E_\mu(-j)$, and (4.17)(c) shows that it can have fixed vectors there only if j is even. We therefore have an orthogonal decomposition

$$E_\mu^{A\cap K} = \sum_{m\in\mathbb{N}} E_\mu^{A\cap K}(m), \qquad E_\mu^{A\cap K}(m) := (E_\mu(2m) + E_\mu(-2m))^{A\cap K} \tag{4.18)(c}$$

We emphasize that this decomposition depends on the choice of simple root α. The simple reflection $s_\alpha \in W$ has a representative $\sigma_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In the Weyl group action $\tau(\mu)$, it follows easily that

$$\tau(\mu)(s_\alpha) = (-1)^m \quad \text{on} \quad (E_\mu^{A\cap K}(m))^* \tag{4.18)(d}$$

Using this same decomposition, one can compute that $a_{s_\alpha}(\mu,\nu)$ is the endomorphism of $(E_\mu^{A\cap K})^*$ which respects the decomposition (4.18)(c), and acts on $(E_\mu^{A\cap K}(m))^*$ by the scalar

$$\prod_{0<j\leq|m|} \frac{j - \langle\nu,\check\alpha\rangle}{j + \langle\nu,\check\alpha\rangle} \tag{4.18)(e}$$

The computation amounts to some formal use of Frobenius reciprocity and induction by stages, and finally a serious calculation in $SL(2,\mathbb{R})$.

In the p-adic case, the operators $a_{s_\alpha}(\tau,\nu)$ were defined directly in terms of the action of $\tau(s_\alpha)$ (cf. (4.12)(c)). In the real case, we use a more subtle structure to compute both the action of $\tau(\mu)(s_\alpha)$ and the operator $a_{s_\alpha}(\mu,\nu)$. In general it is impossible to recover the intertwining operator (which sees all of the different values of the non-negative integer m) from the Weyl group action (which sees only the parity of m). But if m is only allowed to take the values 0 and 1, then we *can* get the intertwining operator from the Weyl group action.

Definition 4.19. An irreducible representation (μ, V_μ) of K is called *petite* if for every simple root α, the representation $\mu \circ \phi_\alpha$ of $SO(2)$ does *not* contain the character χ_{2j} for $|j| \geq 2$.

Theorem 4.20. *Suppose G is a split real groups, and (μ, V_μ) is a petite irreducible representation of K. For each simple root α, the intertwining factor of (4.18)(e) is*

$$a_{s_\alpha}(\mu,\nu) = \begin{cases} 1 & \text{on the } 1 \text{ eigenspace of } \tau(\mu)(s_\alpha), \\ \frac{1-\langle \nu, \check{\alpha}\rangle}{1+\langle \nu, \check{\alpha}\rangle} & \text{on the } -1 \text{ eigenspace of } \tau(\mu)(s_\alpha). \end{cases}$$

The full intertwining operator $a(\mu,\nu)$ is therefore given by the p-adic operator

$$a(\mu,\nu) = a_{w_0}(\tau(\mu),\nu).$$

The first formula is just (4.18)(d) and (4.18)(e), together with the constraint $m = 0$ or 1 coming from Definition 4.19. The second follows from the factorization of the intertwining operator (written in (4.9) in the p-adic case, and left unwritten in the real case).

Theorem 4.20 shows that the signature of the invariant Hermitian form on a petite K-type μ of a spherical representation $\overline{X}(\nu)$ is equal to the signature on a certain K-type of the corresponding spherical representation of the split p-adic group with the same root system. (A little more precisely, we need to take a sum of K-types in the p-adic case, because the Weyl group representation $\tau(\mu)$ may be reducible.)

Here is an example. Let μ_r be the adjoint action of K on $\mathfrak{g}/\mathfrak{k}$. (The subscript r stands for "reflection.") If G is simple of adjoint type, then μ_r is irreducible, and

$$(\mathfrak{g}/\mathfrak{k})^{A\cap K} \simeq \mathfrak{a};$$

the corresponding representation $\tau(\mu_r)$ is the reflection representation of W on $\mathfrak{a}$. The representation μ_r is always petite, so Theorem 4.20 calculates the intertwining operator $a(\mu_r,\nu)$.

In [B1], Barbasch shows that for G a split real classical group, a spherical principal series representation $\overline{X}(\nu)$ is unitary if and only the Hermitian form is positive on each petite K-type; that is, if and only if the operators $a(\mu,\nu)$ are positive semidefinite for each petite μ. The proof does not really explain why this should be so: what he does is make an explicit list of all ν satisfying these (obviously necessary) conditions for unitarity, and then proves by other means that the representations on this list are all unitary. In fact Barbasch uses a subset of the petite K-types that he calls "relevant"; the problems below could be complicated a bit to include the goal of finding a general definition of "relevant."

Problem 4.21. For every simple split real group G, find all petite representations μ of K, and calculate the corresponding Weyl group representations $\tau(\mu)$ (cf. (4.16)). Find an *a priori* characterization of this set of Weyl group representations.

The first part of this problem (listing petite K-types) is fairly easy to do by hand, and the second part is not too difficult. The last asks for some mathematical insight, and it is difficult to predict how easy or hard it might be. At least it is clear that these Weyl group representations constitute a rather small part of $\widehat{W}$. For p-adic classical groups as well, unitarity of spherical representations is characterized by the semidefiniteness of $a_{w_0}(\tau,\nu)$ for this small collection of Weyl group representations. This suggests some additional problems.

Problem 4.22. In the setting of Problem 4.14, find a minimal set of Weyl group representations $\{\tau_i\}$ so that positive semidefiniteness of $a_{w_0}(\tau_i,\nu)$ for all i guarantees semidefiniteness of $a_{w_0}(\tau,\nu)$ for all τ. Find some general reason for semidefiniteness of certain $a_{w_0}(\tau_j,\nu)$ to imply semidefiniteness of another $a_{w_0}(\tau,\nu)$.

This problem should be related to the last part of Problem 4.21: once we know which τ really matter, we would like to know *a priori* that they are of the form $\tau(\mu)$ for petite μ. For the symmetric group S_n, one minimal set corresponds to the $[n/2]$ partitions of n into exactly two parts.

For unitary representations we are interested only in semidefiniteness; but once one starts computing signatures, it is tempting to continue as far as possible. In light of the example after Theorem 4.20, one natural place to begin is

Problem 4.23. Find a closed form description of the signature of the Hermitian operator $a_{w_0}(\text{reflection}, \nu)$, for any dominant ν such that $w_0\nu = -\nu$.

This matrix has size equal to the rank of G, and is relatively easy to compute; so experimental data are easy to get. The rank of the matrix has been computed by Jesper Bang-Jensen about fifteen years ago.

5. General unitary representations for real groups

As discussed at the end of Section 4 the spherical case is typical of a finite number of cases necessary to compute the unitary dual of a real group G. Here we will discuss the new issues which arise in the more general case.

Fix a rational Levi subgroup L of G, and consider an irreducible representation $X(L, \nu)$. One should think of this being a case for which there is no reduction to a smaller group as in Theorem 3.14, although this isn't necessary.

The main issue is the problem of computing the Hermitian form on a finite set of K–types of X. Such a Hermitian form is computed via intertwining operators, as in the spherical case, but which are computationally much more difficult to understand. What is needed is a computational technique to write this operators, at least for rational values of ν, as an explicit Hermitian matrix with rational entries.

One possible approach is the step algebra of Jouko Mickelsson, as developed by van den Hombergh and Zhelobenko (see [Z]). What one can hope for from this machinery is something like this. Fix a series of standard representations $X_q(\delta, \nu)$, with lowest K-type μ_0, and any other fixed representation μ of K. The step algebra should provide (an algorithm to write down) a finite collection of elements $Z_1, \dots, Z_m$ in the enveloping algebra, each of which carries highest weight vectors for μ_0 to highest weight vectors for μ. If v_0 is a non-zero highest weight vector for the μ_0 K-type of $X_q(\delta, \nu)$, then the m vectors $Z_i v_0$ should span the highest weight space of the μ K-type (for all ν). Computing the Hermitian form on the μ K-type amounts to computing the $m \times m$ matrix $\langle Z_i v_0, Z_j v_0\rangle$, and this we know something about. If the elements Z_i are in some sense rational, then the matrix should have entries that are rational polynomials in ν. (The denominators that appeared in formulas like (4.18)(e) disappear because we are using a different basis for the K-types.)

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