# Assigning representation parameters to `atlas` `block` output*

Annegret Paul

July 2008

The goal of these notes is to explain how to attach representation parameters ($\Lambda$- and $\Gamma$-data, which are characters of tori) to the `block` output of `atlas`, for blocks associated to representations with (regular) infinitesimal character in the translation family of $\rho$, and to do this explicitly in a variety of examples. The `block` output lines are labeled with pairs of numbers $(i, j)$ which correspond to pairs $(x, y)$ as described in [2] and [4], to which correspond representations as defined in [6]. Although a pair $(x, y)$ determines a standard module and an irreducible representation uniquely, the `block` output only contains sufficient information to make an assignment up to outer automorphisms of $G(\mathbb{R})$, and, if $G(\mathbb{R})$ is disconnected, up to tensoring by a character which is trivial on the identity component. In addition, the user will have his or her own realization of the group $G(\mathbb{R})$ in mind, with a choice of representatives of Cartan subgroups, and a choice of isomorphisms of those CSG's with abstract tori. The parameterization of representations will depend on those choices. For example, if we realize $SL(2, \mathbb{R})$ as the set of $2 \times 2$ real matrices of determinant 1, and the compact Cartan as

$$\left\{ k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \right\} \simeq S^1,$$

then the two natural choices of isomorphism

$$k_\theta \mapsto e^{i\theta} \text{ and } k_\theta \mapsto e^{-i\theta} \tag{1}$$

lead to different parametrizations of representations of $SL(2, \mathbb{R})$; making a different choice amounts to switching the holomorphic and antiholomorphic discrete series. It is important to be aware of which choices can be made freely, and which are dependent on other decisions and need to be made consistently. For example, once the choice of

*This material is based upon work supported by the National Science Foundation under Grant No. 0554278.

isomorphism (1) is made, the isomorphism of the split Cartan with $\mathbb{R}^\times$ is then uniquely determined via a Cayley transform.

We will fix the real form $G(\mathbb{R})$ of the group $G$ and start with the "large" block, i. e., the block containing the trivial representation of $G(\mathbb{R})$. Once we have assigned characters of Cartan subgroups to all lines of the output, the assignments in the smaller blocks should (in principle) be uniquely determined by choices we have already made. However, in many cases, `atlas` does not provide enough information to deduce what they are; for example, it would be important to know how the $y$-parameters corresponding to the second entries for pairs $(i, j)$ in two different blocks are related to each other; that information is not available, so we have to make new choices. Moreover, `atlas` often creates identical looking blocks (corresponding to isomorphic strong real forms) only once, although the blocks correspond to nonequivalent representations of $G(\mathbb{R})$. (Consider for example the blocks $PSL(2, \mathbb{R}) \times SU(2, 0)$ and $PSL(2, \mathbb{R}) \times SU(0, 2)$ which give two principal series of $PSL(2, \mathbb{R})$, differing by tensoring with a nontrivial character.)

In light of these ambiguities, we reformulate the goal of these notes as follows: we explain how to find an "isomorphism" between the set of representation parameters for a block $\mathcal{B}$ with the set of `atlas block` parameters for this block; i. e., a bijection such that the nature of the simple roots, the cross actions, and the Cayley transforms match up. In practice, the user does not need to have a complete set of parameters (and understanding of which sets of parameters give equivalent representations) in front of him or her; once one `block` line is assigned to a parameter, the algorithm outlined produces a set of parameters for the whole block. However, it is necessary to understand (have in mind a model of) the appropriate complex torus and root system.

There are no proofs in these notes; they will appear elsewhere [4], [5], [9], [3], [6].

# 1 Attaching characters step by step

We start with the "large" block; by this we mean the block for any real form $G(\mathbb{R})$ and the quasisplit form of $G^\vee$; this is the block that contains the trivial representation of $G(\mathbb{R})$. In this large block, the theory of the character formula of the trivial representation described in [9] helps us to nail things down, without having to calculate the parity of real roots, etc.

The "$\Lambda$-data" consist of a Cartan subgroup (up to conjugacy) $H(\mathbb{R})$, a genuine character $\Lambda$ of the $\rho$-cover of $H(\mathbb{R})$, and a choice $\Phi_R^+$ of positive real roots. As explained in [9], and in more detail in [8], we can specify $\Lambda$ by a pair $(\lambda, \kappa)$, where $\lambda = d\Lambda$ and $\kappa$ is an algebraic character which agrees with $\Lambda$ on the compact part of $H(\mathbb{R})$.

The "$\Gamma$-data" consist of a Cartan subgroup $H(\mathbb{R})$ (the same $H(\mathbb{R})$ as above), a character $\Gamma$ of $H(\mathbb{R})$ (given by a pair $(\gamma, \xi)$ analogous to the above), and a linear functional $\overline{\gamma}$ satisfying

$$d\Gamma = \overline{\gamma} + \rho_i - 2\rho_{i,c}.$$

Here $\rho_i$ and $\rho_{i,c}$ are half sums of the imaginary and compact roots of a positive root system, respectively. We always have $\overline{\gamma} = \lambda$, so we use this latter notation.

Recall that the data parametrize both a standard module $X$ and an irreducible representation $\pi$ which is the unique constituent of $X$ which contains its lowest $K$-types. We can construct this standard module using real parabolic induction as follows: Write

$$H(\mathbb{R}) = TA,$$

the decomposition of $H(\mathbb{R})$ into a compact part and a vector group, and

$$M = Cent_{G(\mathbb{R})}(A) = M'A. \tag{2}$$

Then $M$ is the Levi factor of a cuspidal parabolic subgroup $P = MN$ of $G(\mathbb{R})$, and

$$X = Ind_P^{G(\mathbb{R})}(\sigma \otimes \nu), \tag{3}$$

where $\sigma$ is a discrete series representation of $M'$, and $\nu$ is the character of $A$ obtained by the restriction of $\Gamma$. The restriction of $\Gamma$ to $T$ is a highest weight of a lowest $M \cap K$-type of $\sigma$. In Section 12 we address briefly how to determine the group $M$.

We now outline a step by step process to determine these parameters associated to a large block. In the subsequent sections, we explain each step and perform it on the example of $Sp(4, \mathbb{R})$.

**Step 1** Print the `block` and the `cartan` output for $G(\mathbb{R})$. (Section 2)

**Step 2** Print the `kgb` output for $G^\vee(\mathbb{R})$, and choose the basepoints. (Section 3)

**Step 3** Choose your favorite parametrization of the root system. Check how `atlas` numbers the simple roots. (Section 4)

**Step 4** Match up a representation attached to the fundamental Cartan #0. (Section 5)

**Step 5** Use simple cross actions to obtain the parameters for all other representations attached to the same Cartan subgroup. (Section 7)

**Step 6** Starting from a basepoint representation, perform a Cayley transform through a simple imaginary noncompact root to obtain the parameters for a representation attached to more split Cartan subgroup. (Section 8)

**More steps** Repeat the previous two steps until done.

## 2 The `block` output

Let's look at the information the **block** output of **atlas** gives us. Each line of the block corresponds to a representation of a given real form of our chosen group $G$. The large block is the one for which the real form of the dual group $G^\vee$ is quasisplit. The representations in the large block have infinitesimal character (in the translation familiy of) $\rho$. Below is the output for the example $Sp(4,\mathbb{R}) \times SO(3,2)$:

```
0   (0,6):   0  0  [i1,i1]  1   2   (6,*)    (4,*)
1   (1,6):   0  0  [i1,i1]  0   3   (6,*)    (5,*)
2   (2,6):   0  0  [ic,i1]  2   0   (*,*)    (4,*)
3   (3,6):   0  0  [ic,i1]  3   1   (*,*)    (5,*)
4   (4,4):   1  2  [C+,r1]  8   4   (*,*)    (0,2)   2
5   (5,4):   1  2  [C+,r1]  9   5   (*,*)    (1,3)   2
6   (6,5):   1  1  [r1,C+]  6   7   (0,1)    (*,*)   1
7   (7,2):   2  1  [i2,C-]  7   6   (10,11)  (*,*)   2,1,2
8   (8,3):   2  2  [C-,i1]  4   9   (*,*)    (10,*)  1,2,1
9   (9,3):   2  2  [C-,i1]  5   8   (*,*)    (10,*)  1,2,1
10  (10,0):  3  3  [r2,r1]  11  11  (7,*)    (8,9)   1,2,1,2
11  (10,1):  3  3  [r2,rn]  10  10  (7,*)    (*,*)   1,2,1,2
```
(4)

The first column numbers the representations, starting at 0. The second column contains a pair of numbers; these correspond to the lines in the kgb outputs of the two groups. This is the pair $(x, y) \in \mathcal{Z}$ which determines the representation. The third column gives the length of the twisted involution parameter. The fourth column gives the conjugacy class of Cartan subgroups that the representation is attached to. The numbering is the one given by the **cartan** output of **atlas**. Here is that output for our example $Sp(4,\mathbb{R})$:

```
Cartan #0:
split:  0; compact:  2; complex:  0
canonical twisted involution:
twisted involution orbit size:  1; fiber rank:  2; #X_r:  4
imaginary root system:  B2
real root system is empty
complex factor is empty
real form #2:  [0,1] (2)
real form #1:  [2] (1)
real form #0:  [3] (1)
```

```
Cartan #1:
split:  0; compact:  0; complex:  1
canonical twisted involution:  2,1,2
twisted involution orbit size:  2; fiber rank:  0; #X_r:  2
imaginary root system:  A1
real root system:  A1
complex factor is empty
real form #2:  [0] (1)
real form #1:  [1] (1)


Cartan #2:
split:  1; compact:  1; complex:  0
canonical twisted involution:  1,2,1
twisted involution orbit size:  2; fiber rank:  1; #X_r:  4
imaginary root system:  A1
real root system:  A1
complex factor is empty
real form #2:  [0] (1)


Cartan #3:
split:  2; compact:  0; complex:  0
canonical twisted involution:  1,2,1,2
twisted involution orbit size:  1; fiber rank:  0; #X_r:  1
imaginary root system is empty
real root system:  B2
complex factor is empty
real form #2:  [0] (1)
```

For example, representations 0 through 3 are attached to the compact Cartan #0, hence discrete series representations. Representations 10 and 11 are attached to Cartan #3, which is the split Cartan, hence they are principal series. Now 6 and 7 are attached to the complex Cartan #1, and 4, 5, 8, and 9 to the mixed Cartan #2, which has a compact and a split factor.

The symbols in square brackets refer to the simple roots. Symbols which may occur are

ic=imaginary compact
i1=imaginary noncompact type 1

i2=imaginary noncompact type 2
rn=real non-parity root
r1=real parity root type 1
r2=real parity root type 2
C$^+$,C$^-$=complex roots

In our example, for representation 0 both simple roots are imaginary noncompact, so this is a large discrete series.

The next r columns (r=rank of the group) give the cross actions through the simple roots; the following r columns give the Cayley transforms through the simple roots. For example, from the parameters of representation 0, cross action through the first simple root $\alpha_1$ yields the parameters for representation 1, and Cayley transform through $\alpha_1$ leads to the parameters for 6.

The last entry of the `block` output is the twisted involution which gives the Cartan involution on the torus. The numbers stand for the corresponding simple reflections, and we must compose this with our fundamental involution (coming from the outer automorphism fixing the pinning). In the same rank case (as in our example), this is trivial, so 1,2,1, for instance, stands for $s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$.

# 3 Choosing basepoints

Here is the `kgb` output for $SO(3,2)$:

```
0:  0  0  [n,n]  1  0  2  3
1:  0  0  [n,c]  0  1  2  *
2:  1  2  [r,C]  2  5  *  *  1
3:  1  1  [C,r]  4  3  *  *  2
4:  2  1  [C,n]  3  4  *  6  1,2,1
5:  2  2  [n,C]  5  2  6  *  2,1,2
6:  3  3  [r,r]  6  6  *  *  2,1,2,1
```
(5)

The kgb output lists the kgb orbits with numbers starting at 0. The next two columns are as in the `block` output, length and conjugacy class of Cartan. The symbols in the square brackets describe the simple roots; this time with a little less detail. The possible symbols are
n=imaginary noncompact
c=imaginary compact
r=real
C=complex

Then we have, as for the `block` output, cross actions, Cayley transforms, and twisted involution.

The basepoints consists of one kgb orbit in each fiber of the space, i. e., for each twisted involution. The main basepoint is the kgb orbit (on the dual side) which gives the trivial representation of the quasiplit form of $G$. It is a closed orbit (length 0), corresponding to a large fundamental series of $G^\vee(\mathbb{R})$. In the same rank case, it satisfies that all simple roots are imaginary noncompact. Once we have chosen the fundamental base point, the others are obtained by performing Cayley transforms through simple imaginary roots and cross actions through simple complex roots. Recall from [9] that the `block` elements (i,j) with j a basepoint give the standard representations occurring in the character formula of the trivial representation. They are also precisely those for which $\Lambda$ is an algebraic character.

In our example, the fundamental basepoint is 0 (we have no choice here); then we get 2 and 3 by Cayley transform through $\alpha_1$ and $\alpha_2$. Cross action through $\alpha_2$ leads from 2 to 5, and cross action through $\alpha_1$ gets us from 3 to 4. Another Cayley transform brings us from either 4 or 5 to 6. So the base points are 0, 2, 3, 4, 5, and 6.

(Fokko has written some software to pick the basepoints**...**maybe we can reinstall it...)

It may be useful to mark the "basepoint representations" in your `block` output. They are the ones that will have $\Lambda$-data satisfying $\kappa = \lambda$:

```
0   (0,6):   0  0  [i1,i1]  1   2   (6,*)    (4,*)            B
1   (1,6):   0  0  [i1,i1]  0   3   (6,*)    (5,*)            B
2   (2,6):   0  0  [ic,i1]  2   0   (*,*)    (4,*)            B
3   (3,6):   0  0  [ic,i1]  3   1   (*,*)    (5,*)            B
4   (4,4):   1  2  [C+,r1]  8   4   (*,*)    (0,2)   2        B
5   (5,4):   1  2  [C+,r1]  9   5   (*,*)    (1,3)   2        B
6   (6,5):   1  1  [r1,C+]  6   7   (0,1)    (*,*)   1        B
7   (7,2):   2  1  [i2,C-]  7   6   (10,11)  (*,*)   2,1,2    B
8   (8,3):   2  2  [C-,i1]  4   9   (*,*)    (10,*)  1,2,1    B
9   (9,3):   2  2  [C-,i1]  5   8   (*,*)    (10,*)  1,2,1    B
10  (10,0):  3  3  [r2,r1]  11  11  (7,*)    (8,9)   1,2,1,2  B
11  (10,1):  3  3  [r2,rn]  10  10  (7,*)    (*,*)   1,2,1,2
```

(6)

# 4 The root system

We can write down a (complex) torus and root system. In our example, we choose

$$H = \left(\mathbb{C}^\times\right)^2 = \left\{(z_1, z_2) : z_i = r_i e^{i\theta_i} \in \mathbb{C}^\times\right\},$$

and

$$\Psi = \{\pm(e_1 + e_2), \pm(e_1 - e_2), \pm 2e_1, \pm 2e_2\}.$$

`atlas` chooses $\alpha_1$ to be the short simple root, and $\alpha_2$ the long one. The list of the `atlas` numberings will be added to the Root Systems explorer on the Atlas website; it is essentially the Bourbaki numbering.

# 5 Where to start

## 5.1 Equal rank case

In the equal rank case, the fundamental Cartan subgroup is compact, corresponding to the trivial involution of the torus, $\theta = 1$. With respect to this Cartan, all roots are imaginary, and there may be a standard choice of compact roots; in our example

$$\begin{aligned} \Psi_{i,c} &= \{\pm(e_1 - e_2)\}\,, \\ \Psi_{i,n} &= \{\pm(e_1 + e_2), \pm 2e_1, \pm 2e_2\}\,. \end{aligned}$$

We start by choosing a discrete series; write down the Harish-Chandra parameter of a discrete series with infinitesimal character $\rho$; this will be the parameter $\lambda$ for the corresponding $\Lambda$-data. In the quasisplit case, a large discrete series is a good choice here, but any other will work as well. The element $\lambda$ determines simple roots $\alpha_1$, $\alpha_2$,..., $\alpha_r$, which are either compact or noncompact. Choose a `block` entry corresponding to the Cartan #0, and with simple roots of this type, in the same order. There may be more than one. For example, for

$$\lambda = (2, 1),$$

we have

$$\begin{aligned} \alpha_1 &= e_1 - e_2 \\ \alpha_2 &= 2e_2. \end{aligned}$$

Since $\alpha_1$ is compact and $\alpha_2$ is noncompact, we look for [ic,i1] or [ic,i2] in the first 4 lines of the `block` output. The possibilities are lines 2 or 3. The two are indistinguishable; the representations differ by an outer automorphism of $Sp(4, \mathbb{R})$. Choosing one of them amounts to choosing a parametrization. Let's choose a large discrete series instead.

$$\begin{aligned} \lambda &= (2, -1) = \kappa \\ \alpha_1 &= e_1 + e_2 \\ \alpha_2 &= -2e_2 \\ \Psi^+ &= \{e_1 + e_2, -2e_2, e_1 - e_2, 2e_1\} \end{aligned}$$

Now both simple roots are noncompact. We assign this parameter to representation 0 in the `block` output.

As mentioned above, the Harish-Chandra parameter will be $\lambda$ for the $\Lambda$ parameter. This is also necessarily a basepoint representation, so we have $\kappa = \lambda$. There are no real roots, so $\Phi_R^+ = \varnothing$ (see Remark 1). We are also keeping track of our other positive roots. For the purpose of computing the $\Gamma$-data, we should also note the compact and noncompact roots:

$$\begin{aligned} \Psi_{i,c}^+ &= \{e_1 - e_2\}\,; \\ \Psi_{i,n}^+ &= \{2e_1, e_1 + e_1, -2e_2\}\,. \end{aligned}$$

The character $\Lambda$ of $H_0 = H(\mathbb{R}) = \left\{(e^{i\theta_1}, e^{i\theta_2})\right\}$ is given by

$$(e^{i\theta_1}, e^{i\theta_2}) \longmapsto e^{i(2\theta_1 - \theta_2)}.$$

To get the $\Gamma$-data of this representation, we recall that $\Gamma$ can be obtained from $\Lambda$ by

$$\begin{aligned} \gamma &= \lambda + \rho_i - 2\rho_{i,c}; \\ \xi &= \kappa + \rho_i - 2\rho_{i,c} + \rho_R + \rho_{cx}^\circ. \end{aligned} \tag{7}$$

Here $\rho_i$ and $\rho_{i,c}$ are the $\rho$ shifts corresponding to our working system $\Psi^+$, $\rho_R$ is one half the sum of the roots in $\Phi_R^+$, and $\rho_{cx}^\circ$ is a certain half sum of complex roots described in section 6 below.

In our example, $\rho_i = (2, -1)$, $2\rho_{i,c} = (1, -1)$, and $\rho_R = 0 = \rho_{cx}^\circ$, so we have

$$\gamma = \xi = (3, -1).$$

Recall that this is the highest weight of the lowest $K$-type of the representation.

## 5.2 Unequal rank case

In the "split" inner class (if distinct from the compact one), the fundamental involution of the torus is not trivial. In order to assign parameters to one of the lines, we need to know what it is. In this case, we know that the involution of the split Cartan is inversion, and we can work back from there. Let's look at the example $SL(3, \mathbb{R})$. Below is the `block` output for the large block.

```
0  (0,5):  0  0  [C+,C+]  2  1  (*,*)  (*,*)
1  (1,4)   1  0  [i2,C-]  1  0  (3,4)  (*,*)  2,1
2  (2,3)   1  0  [C-,i2]  0  2  (*,*)  (3,5)  1,2
3  (3,0)   2  1  [r2,r2]  4  5  (1,*)  (2,*)  1,2,1
4  (3,1)   2  1  [r2,rn]  3  4  (1,*)  (*,*)  1,2,1
5  (3,2)   2  1  [rn,r2]  5  3  (*,*)  (2,*)  1,2,1
```
(8)

The `cartan` command tells us that the group has two conjugacy classes of Cartan subgroups, one complex and one split. So representation #0 must be attached to a Cartan which is isomorphic to $\mathbb{C}^\times$. Let's work out a realization of it.

In type $A_2$, it is convenient and standard to write

$$H = \left\{(z_1, z_2, z_3) \in \left(\mathbb{C}^\times\right)^3 : z_1 z_2 z_3 = 1\right\} \simeq \left(\mathbb{C}^\times\right)^2.$$

Then the roots are

$$\Psi = \{\pm(e_i - e_j) : 1 \le i < j \le 3\}.$$

Choose our simple roots to be

$$\alpha_1 = e_1 - e_2 \text{ and } \alpha_2 = e_2 - e_3.$$

The twisted involution for the split Cartan is obtained from the fundamental involution by composition with $s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$; this is permutation of the first and third coordinates. So the fundamental involution $\theta = \theta_0$ must be given by

$$\theta_0(z_1, z_2, z_3) = (z_3^{-1}, z_2^{-1}, z_1^{-1}) \tag{9}$$

for $(z_1, z_2, z_3) \in H$. Now it is easy to see that

$$\begin{aligned} H^\theta &= \left\{(z, 1, z^{-1}) : z \in \mathbb{C}^\times\right\}, \text{ and} \\ H^{-\theta} &= \left\{(z, w, z) : z^2 w = 1\right\} = \left\{(z, z^{-2}, z)\right\}. \end{aligned}$$

So we get

$$H_0 = H(\mathbb{R}) = \left\{h_{r,\varphi} = (re^{i\varphi}, r^{-2}, re^{-i\varphi}) : r \in \mathbb{R}_+^\times, \varphi \in \mathbb{R}\right\} \simeq \mathbb{C}^\times. \tag{10}$$

We can now check that $\alpha_1$ and $\alpha_2$ are both complex (since they take complex values on the real Lie algebra of $H_0$), so this indeed matches with `block` line 0. There are no real roots. With our choice of simple roots, $\rho = (1, 0, -1)$, so representation #0 has the following $\Lambda$-data:

$$\begin{aligned} & H(\mathbb{R}) \text{ given by (10)} \\ \lambda &= (1, 0, -1) = \kappa \\ \Phi_R^+ &= \varnothing \end{aligned}$$

Notice that the character $\Lambda$ of $H_0$ is given by

$$\Lambda(h_{r,\varphi}) = re^{i\varphi}(r^{-2})^0 r^{-1} e^{i\varphi} = e^{2i\varphi}.$$

To compute the $\Gamma$-data, we need to know how to compute the complex $\rho$-shift $\rho_{cx}^\circ$.

**Remark 1** *In general, the real root system $\Phi_R^+$ will be given by the negative real roots of our system $\Psi$ which we are keeping track of along the way:*

$$\Phi_R^+ = \left\{-\alpha : \alpha \in \Psi_R^+\right\} = \{\alpha \in \Psi_R : \langle \lambda, \alpha^\vee \rangle < 0\}\,. \tag{11}$$

*Notice that this is uniquely determined by $\lambda$, so that the $\Lambda$-data may be specified by giving just $H(\mathbb{R})$, $\lambda$, and $\kappa$.*

# 6 The complex shift $\rho_{cx}^\circ$

The construction of a representation with certain $\Lambda$-data involves choosing a positive roots system $\Delta^+$ which contains the imaginary roots positive on $\lambda$, the real roots $\Phi_R^+$, and a set of positive complex roots satisfying

$$\beta \text{ complex and } \beta > 0 \Longrightarrow \theta\beta < 0. \tag{12}$$

This system is in general not the one determined by $\lambda$. The shift $\rho_{cx}^\circ$ in the conversion formula (7) is then one half of the sum of these positive complex roots in $\Delta^+$. Fortunately, the character $\Gamma$ is independent of the choice of the set of positive complex roots, subject to condition (12), i. e., any two such choices will result in values of $\rho_{cx}^\circ$ which differ by an element of $(1-\theta)X^*(H)$ (hence have the same restriction to the compact part of the torus), so we are free to choose it without having to worry whether it is indeed a subset of some positive root system.

For the fundamental Cartan in our example $SL(3,\mathbb{R})$, the complex roots are

$$\pm(e_1 - e_2) \text{ and } \pm(e_2 - e_3),$$

and from (9) we have

$$\theta_0(e_1 - e_2) = e_2 - e_3.$$

Therefore, the set $\{e_1 - e_2, -e_2 + e_3\}$ satisfies (12). So we can choose

$$\rho_{cx}^\circ = \left(\frac{1}{2}, -1, \frac{1}{2}\right).$$

To finish the calculation of $\Gamma$-data, we have $\rho_R = 0$, and since $e_1 - e_3$ is imaginary noncompact (we'll explain how to see this in Section 7.2), $\rho_{i,c} = 0$, and $\rho_i = \left(\frac{1}{2}, 0, -\frac{1}{2}\right)$. So

$$\begin{aligned} \gamma &= \lambda + \rho_i - 2\rho_{i,c} = \left(\frac{3}{2}, 0, -\frac{3}{2}\right), \\ \xi &= \kappa + \rho_i - 2\rho_{i,c} + \rho_R + \rho_{cx}^\circ = (2, -1, -1)\,. \end{aligned} \tag{13}$$

The character $\Gamma$ is therefore given by

$$\Gamma(h_{r,\varphi}) = e^{3i\varphi}.$$

Notice that since $H(\mathbb{R})$ is connected, the character $\Gamma$ is uniquely determined by its differential $\gamma$. Indeed, $\gamma$ and $\xi$ agree on $H^\theta = \{(z, 1, z^{-1}) : z \in \mathbb{C}^\times\}$, so that we could have chosen $\xi = \gamma$.

We will finish the example $SL(3, \mathbb{R})$ in Section 13.3.

# 7 Cross actions

Once we have the $\Lambda$-data for a representation associated to a given conjugacy class of Cartan subgroups, we can obtain those for the other representations attached to the same Cartan subgroup by applying simple cross actions to the parameters. The `block` output tells us which cross action takes us to which representation. For cross actions through imaginary or complex simple roots, we essentially apply the corresponding root reflection to $\lambda$ and $\Psi^+$ to obtain our new parameters; for cross action through real roots, only $\kappa$ changes.

## 7.1 Imaginary cross actions

Cross actions through simple noncompact imaginary roots take us to other representations attached to the same twisted involution. For example, in the equal rank case, we can derive the parameters for all discrete series from our starting point.

More precisely, suppose we have $\Lambda$-data

$$\left(H(\mathbb{R}), \lambda, \kappa, \Phi_R^+, \Psi^+\right) \tag{14}$$

(we include our "working root system" $\Psi^+$ as part of the data, for convenience). If $\alpha$ is a simple imaginary noncompact root, then cross action through $\alpha$ takes us to the $\Lambda$-data

$$\left(H(\mathbb{R}), s_\alpha\lambda, s_\alpha\lambda + (\kappa - \lambda), \Phi_R^+, s_\alpha\Psi^+\right). \tag{15}$$

The nature of the roots (imaginary/complex/real, compact/noncompact) does not change under cross actions. If $\Lambda$-data (14) are attached to representation $i$ in the `block` output, and the output indicates that cross action through $\alpha$ takes us to line $j$, then $\Lambda$-data (15) are attached to representation $j$.

We can now easily write down the $\Lambda$-data for the remaining three discrete series representations of $Sp(4, \mathbb{R})$. From #0, cross action through $\alpha_1 = e_1 + e_2$ takes us to representation #1:

$$\lambda = s_{e_1+e_2}(2, -1) = (1, -2) = \kappa \tag{16}$$

Our new simple roots are

$$\alpha_1 = -e_1 - e_2, \quad (17)$$
$$\alpha_2 = 2e_1. \quad (18)$$

The new $\rho$ shifts are $\rho_i = (2, -1)$ and $2\rho_{i,c} = (1, -1)$, so we have (the highest weight of the lowest $K$-type)

$$\gamma = \xi = (3, -1). \quad (19)$$

If we perform cross action (from #0) through $\alpha_2 = -2e_2$ instead, we get the data for representation #2:

$$\lambda = (2, 1) = \kappa \quad (20)$$
$$\alpha_1 = e_1 - e_2 \quad (21)$$
$$\alpha_2 = 2e_2 \quad (22)$$
$$\gamma = \xi = (3, 3). \quad (23)$$

We get to #3 from #1 by cross action through $\alpha_2 = 2e_1$:

$$\lambda = (-1, -2) \quad (24)$$
$$\alpha_1 = e_1 - e_2 \quad (25)$$
$$\alpha_2 = -2e_1 \quad (26)$$
$$\gamma = \xi = (-3, -3). \quad (27)$$

To keep this information organized, we make a table of what we have so far.

| # | $\theta$ | CSG | $\alpha_1$ | $\alpha_2$ | $\lambda$ | $\kappa$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | $e_1 + e_2$ | $-2e_2$ | $(2, -1)$ | $(2, -1)$ |
| 1 | 1 | 0 | $-e_1 - e_2$ | $2e_1$ | $(1, -2)$ | $(1, -2)$ |
| 2 | 1 | 0 | $e_1 - e_2$ | $2e_2$ | $(2, 1)$ | $(2, 1)$ |
| 3 | 1 | 0 | $e_1 - e_2$ | $-2e_1$ | $(-1, -2)$ | $(-1, -2)$ |

(28)

## 7.2 Complex cross actions

Cross actions through simple complex roots take us to the parameters of a representation which is attached to the same conjugacy class of Cartan subgroups, but with different twisted involution. We conjugate things to the same torus, with the same Cartan involution. The formulas for the new $\Lambda$-data is the same as in the imaginary case above, in (15). Notice that the positive real root system does not change. The same is true for the positive imaginary roots. As in the previous case, the nature of the roots remains the same. This allows us to determine the nature of nonsimple roots; choose a complex simple cross action which will turn the nonsimple root in question into a simple one, then we can read the nature off the `block` output. This allowed us in the $SL(3,\mathbb{R})$ example to determine that the imaginary roots were noncompact, since they are simple in representations #1 and #2.

We will perform cross actions through simple complex roots for $Sp(4,\mathbb{R})$ once we have $\Lambda$-data attached to other Cartan subgroups (after performing Cayley transforms).

### 7.3 Real cross actions

As for cross actions through imaginary roots, those through simple real roots take us to representations attached to the same twisted involution. In this case we conjugate everything so that the only part of the data that changes is $\kappa$. More explicitly, cross action through the simple real root $\alpha$ takes the $\Lambda$-data given in (14) to

$$\left(H(\mathbb{R}), \lambda, \lambda + s_\alpha(\kappa - \lambda) + \alpha, \Phi_R^+, \Psi^+\right). \tag{29}$$

# 8 Cayley transforms

Cayley transforms through simple imaginary noncompact roots lead from the parameters of one representation to those of one or two (depending on the type of root) representations attached to a less compact Cartan. The imaginary root $\alpha$ in question becomes real. If the Cayley transform is double-valued, the two parameters in question will be related by a (real) cross action through $\alpha$. (So they will have different $\kappa$ parameters). In order to avoid ambiguities, for each conjugacy class of Cartan subgroups, we perform only one Cayley transform with the resulting parameter attached to that class. All other parameters for this Cartan will then be obtained by cross actions. Moreover, we always choose the data of a basepoint representation for the Cayley transform; one of the resulting representations on the new Cartan will also be a basepoint representation. This way we will be sure to have the correct $\kappa$ parameter (namely, $\kappa = \lambda$).

The first thing to determine is the new Cartan subgroup $H(\mathbb{R})$. This is determined by the new Cartan involution, which is obtained by composition with the root reflection $s_\alpha$. The other data, $\lambda = \kappa$ and $\Psi^+$, will stay unchanged. The nature of the roots changes, and $\Phi_R^+$ becomes larger; in particular, it will contain $-\alpha$.

Let's perform a Cayley transform through $\alpha_1 = e_1 + e_2$ on the parameters of our large discrete series #0. According to the `block` output, this must give us the parameters for representation #6, which is attached to the complex Cartan #1. Call the corresponding involution $\theta_1$. Since $\theta_0$ is trivial, $\theta_1 = s_{e_1+e_2}$, i. e.,

$$\theta_1(z_1, z_2) = (z_2^{-1}, z_1^{-1}). \tag{30}$$

So

$$\begin{aligned} H^{\theta_1} &= \left\{(z, z^{-1}) : z \in \mathbb{C}^\times\right\} &(31)\\ H^{-\theta_1} &= \left\{(z, z) : z \in \mathbb{C}^\times\right\} &(32)\\ H_1 &= H(\mathbb{R}) = \left\{h_{r,\varphi} = (re^{i\varphi}, re^{-i\varphi}) : r \in \mathbb{R}_+^\times, \varphi \in \mathbb{R}\right\} \simeq \mathbb{C}^\times &(33) \end{aligned}$$

We still have

$$\begin{aligned} \lambda &= (2, -1) = \kappa \\ \alpha_1 &= e_1 + e_2 \\ \alpha_2 &= -2e_2 \\ \Psi^+ &= \{e_1 + e_2, -2e_2, e_1 - e_2, 2e_1\} \end{aligned}$$

but now $\alpha_1 = e_1 + e_2$ is real, $2e_1$ and $-2e_2$ are complex, and $e_1 - e_2$ is still imaginary. (One way to determine the nature of these roots is to just check the values on the real Cartan subalgebra.) To see that $e_1 - e_2$ is actually noncompact (type 2), we see what happens to these parameters if we perform a cross action through the complex root $\alpha_2 = -2e_2$; then the root becomes simple, and `atlas` gives us the answer. So we now have

$$\begin{aligned} \Phi_R^+ &= \{-e_1 - e_2\} &(34)\\ \rho_i &= \left(\frac{1}{2}, -\frac{1}{2}\right) &(35)\\ \rho_{i,c} &= 0. &(36) \end{aligned}$$

Since the Cartan is connected, we don't need to compute $\rho_{cx}^\circ$. The character $\Lambda$ is given by

$$\Lambda(h_{r,\varphi}) = re^{3i\varphi}. \tag{37}$$

We can also easily compute $\Gamma$: since $H_1$ is connected, $\gamma = \xi$, and we have

$$\gamma = \left(\frac{5}{2}, -\frac{3}{2}\right), \tag{38}$$

so

$$\Gamma(h_{r,\varphi}) = re^{4i\varphi}. \tag{39}$$

So we have now

| # | $\theta$ | CSG | $\alpha_1$ | $\alpha_2$ | $\lambda$ | $\kappa$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | $e_1 + e_2$ | $-2e_2$ | $(2, -1)$ | $(2, -1)$ |
| 1 | 1 | 0 | $-e_1 - e_2$ | $2e_1$ | $(1, -2)$ | $(1, -2)$ |
| 2 | 1 | 0 | $e_1 - e_2$ | $2e_2$ | $(2, 1)$ | $(2, 1)$ |
| 3 | 1 | 0 | $e_1 - e_2$ | $-2e_1$ | $(-1, -2)$ | $(-1, -2)$ |
| 6 | $s_{e_1+e_2}$ | 1 | $e_1 + e_2$ | $-2e_2$ | $(2, -1)$ | $(2, -1)$ |

(40)

# 9 Finishing the example $Sp(4, \mathbb{R})$

Since we have a set of $\Lambda$-data for CSG #1, we can now apply a complex cross action (through $\alpha_2 = -2e_2$) to obtain the data for the other representation attached to it, #7. We get

$$\lambda = (2, 1) = \kappa \tag{41}$$
$$\alpha_1 = e_1 - e_2 \tag{42}$$
$$\alpha_2 = 2e_2 \tag{43}$$
$$\Phi_R^+ = \{-e_1 - e_2\} \tag{44}$$

The character $\Lambda$ is given by

$$\Lambda(h_{r,\varphi}) = r^3 e^{i\varphi}. \tag{45}$$

We leave the calculation of $\Gamma$ to the diligent reader.

This last representation has a simple noncompact root of type 2, $\alpha_1 = e_1 - e_2$. We perform a Cayley transform to get the parameters for representation #10. (The Cayley transform is double-valued, 10 and 11, but we choose the basepoint representation of the two.) The new Cartan involution is $s_{e_1 - e_2}\theta_1 =$ inversion, so the Cartan is split, $H_3 = (\mathbb{R}^\times)^2$. All roots are real now, and

$$\lambda = (2, 1) = \kappa \tag{46}$$
$$\alpha_1 = e_1 - e_2 \tag{47}$$
$$\alpha_2 = 2e_2 \tag{48}$$
$$\Phi_R^+ = \{-e_1 - e_2, -e_1 + e_2, -2e_1, -2e_2\} \tag{49}$$

The character $\Lambda$ is given by

$$\Lambda(r_1, r_2) = r_1^2 r_2 \text{ for } r_i \in \mathbb{R}^\times. \tag{50}$$

This time the Cartan subgroup is disconnected, so let's compute $\Gamma$. We have

$$\gamma = \lambda + \rho_i - 2\rho_{i,c} = \lambda = (2,1)\,, \tag{51}$$

$$\xi = \kappa + \rho_i - 2\rho_{i,c} + \rho_R + \rho_{cx}^\circ = (2,1) - 0 + (-2,-1) + 0 = (0,0)\,. \tag{52}$$

This means that

$$\Gamma(r_1, r_2) = |r_1|^2\,|r_2|\,. \tag{53}$$

This representation is, of course, the trivial representation, or, the spherical principal series.

Now we only need to perform a real cross action through $\alpha_1 = e_1 - e_2$ to get the data for #11:

$$\lambda = (2,1) = \gamma \tag{54}$$

$$\kappa = \lambda + \alpha_1 = (3,0) \tag{55}$$

$$\xi = (1,-1) \tag{56}$$

$$\alpha_1 = e_1 - e_2 \tag{57}$$

$$\alpha_2 = 2e_2 \tag{58}$$

$$\Phi_R^+ = \{-e_1 - e_2, -e_1 + e_2, -2e_1, -2e_2\} \tag{59}$$

$$\Lambda(r_1, r_2) = |r_1|^2\,|r_2|\,sgn(r_1) \tag{60}$$

$$\Gamma(r_1, r_2) = |r_1|^2\,|r_2|\,sgn(r_1 r_2) \tag{61}$$

It remains to determine the data for the representations attached to the mixed Cartan #2. We can start by performing a Cayley transform through $\alpha_1 = -2e_2$ to the parameters of representation #0 to obtain the data for representation #4. We have

$$\theta_2(z_1, z_2) = (z_1, z_2^{-1}),\ \text{so} \tag{62}$$

$$H_2 = \left\{\left(e^{i\varphi}, r\right) : \varphi \in \mathbb{R}, r \in \mathbb{R}^\times\right\} = S^1 \times \mathbb{R}^\times \tag{63}$$

and

$$\begin{aligned} \lambda &= (2,-1) = \kappa \\ \alpha_1 &= e_1 + e_2 \\ \alpha_2 &= -2e_2 \\ \Phi_R^+ &= \{2e_2\} \\ \rho_i &= (1,0)\,, \rho_{i,c} = 0 \end{aligned}$$

The short roots are complex, with

$$\theta_2(e_1 - e_2) = e_1 + e_2, \tag{64}$$

so we can take

$$\rho_{cx}^{\circ} = \frac{1}{2}\left((e_1 - e_2) + (-e_1 - e_2)\right) = (0, -1)\,. \tag{65}$$

So

$$\begin{aligned} \gamma &= (3, -1) && (66)\\ \xi &= \gamma + \rho_R + \rho_{cx}^{\circ} = \gamma, && (67) \end{aligned}$$

and

$$\begin{aligned} \Lambda\left(e^{i\varphi}, r\right) &= e^{2i\varphi} r^{-1} = e^{2i\varphi} \left|r\right|^{-1} sgn(r) && (68)\\ \Gamma\left(e^{i\varphi}, r\right) &= e^{3i\varphi} r^{-1} = e^{3i\varphi} \left|r\right|^{-1} sgn(r). && (69) \end{aligned}$$

For the remaining three representations, we only compute the $\Lambda$-data. Cross action through $\alpha_1 = e_1 + e_2$ gives us the data for representation #8:

$$\begin{aligned} \lambda &= (1, -2) = \kappa && (70)\\ \alpha_1 &= -e_1 - e_2 && (71)\\ \alpha_2 &= 2e_1 && (72)\\ \Phi_R^+ &= \{2e_2\} && (73)\\ \Lambda\left(e^{i\varphi}, r\right) &= e^{i\varphi} r^{-2} && (74) \end{aligned}$$

From #8, we get #9 by cross action through $\alpha_2 = 2e_1$:

$$\begin{aligned} \lambda &= (-1, -2) = \kappa && (75)\\ \alpha_1 &= e_1 - e_2 && (76)\\ \alpha_2 &= -2e_1 && (77)\\ \Phi_R^+ &= \{2e_2\} && (78)\\ \Lambda\left(e^{i\varphi}, r\right) &= e^{-i\varphi} r^{-2} && (79) \end{aligned}$$

and from #9, we get #5 by cross action through $\alpha_1 = e_1 - e_2$:

$$\begin{aligned} \lambda &= (-2, -1) = \kappa && (80)\\ \alpha_1 &= -e_1 + e_2 && (81)\\ \alpha_2 &= -2e_2 && (82)\\ \Phi_R^+ &= \{2e_2\} && (83)\\ \Lambda\left(e^{i\varphi}, r\right) &= e^{-2i\varphi} r^{-1} && (84) \end{aligned}$$

We are collecting our results in this final table:

| # | $\theta$ | CSG | $\alpha_1$ | $\alpha_2$ | $\lambda$ | $\kappa$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | $e_1 + e_2$ | $-2e_2$ | $(2, -1)$ | $(2, -1)$ |
| 1 | 1 | 0 | $-e_1 - e_2$ | $2e_1$ | $(1, -2)$ | $(1, -2)$ |
| 2 | 1 | 0 | $e_1 - e_2$ | $2e_2$ | $(2, 1)$ | $(2, 1)$ |
| 3 | 1 | 0 | $e_1 - e_2$ | $-2e_1$ | $(-1, -2)$ | $(-1, -2)$ |
| 4 | $s_{2e_2}$ | 2 | $e_1 + e_2$ | $-2e_2$ | $(2, -1)$ | $(2, -1)$ |
| 5 | $s_{2e_2}$ | 2 | $-e_1 + e_2$ | $-2e_2$ | $(-2, -1)$ | $(-2, -1)$ |
| 6 | $s_{e_1+e_2}$ | 1 | $e_1 + e_2$ | $-2e_2$ | $(2, -1)$ | $(2, -1)$ |
| 7 | $s_{e_1+e_2}$ | 1 | $e_1 - e_2$ | $2e_2$ | $(2, 1)$ | $(2, 1)$ |
| 8 | $s_{2e_2}$ | 2 | $-e_1 - e_2$ | $2e_1$ | $(1, -2)$ | $(1, -2)$ |
| 9 | $s_{2e_2}$ | 2 | $e_1 - e_2$ | $-2e_1$ | $(-1, -2)$ | $(-1, -2)$ |
| 10 | inv | 3 | $e_1 - e_2$ | $2e_2$ | $(2, 1)$ | $(2, 1)$ |
| 11 | inv | 3 | $e_1 - e_2$ | $2e_2$ | $(2, 1)$ | $(3, 0)$ |

(85)

**Remark 2** *Data which are conjugate by the real Weyl group parametrize equivalent representations; therefore, the $\Lambda$-data and $\Gamma$-data are not uniquely determined. For example, we can obtain the data for representation #4 by Cayley transform through $\alpha_2$ from discrete series #2 instead of from #0; the resulting character $\Lambda$ would be given by*

$$\lambda = (2, 1) = \kappa$$

*instead, with the Cartan $H_2$ realized exactly as above. Since the root reflection $s_{2e_2}$ belongs to the real Weyl group, these are equivalent parametrizations.*

# 10 Smaller blocks for $Sp(4, \mathbb{R})$

Those familiar with the representation theory of $Sp(4, \mathbb{R})$ will know that we are missing six representations at infinitesimal character $\rho$: they are all attached to disconnected Cartans and non-basepoint representations; i.e., $\kappa$ would be different from $\lambda$. These representations will be found in the two smaller blocks, $Sp(4, \mathbb{R}) \times SO(4, 1)$ and $Sp(4, \mathbb{R}) \times SO(5)$. In general, representations in smaller blocks may be attached to different (translation families of) infinitesimal characters; see Section 13.1.2 for more details. For simply connected $G$, however, there is only one such family, so all blocks give representations at infinitesimal character $\rho$. In these smaller blocks, we don't have basepoints to nail things down, but we can often still assign the $\Lambda$-data correctly. One fact working in our favor in this example is that there are no double-valued Cayley transforms. Here is the `block` output for the first block, $Sp(4,\mathbb{R}) \times SO(4,1)$:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0(4,2): | 1 | 2 | [C+,rn] | 2 | 0 | (*,*) | (*,*) | 2 |
| 1(5,2): | 1 | 2 | [C+,rn] | 3 | 1 | (*,*) | (*,*) | 2 |
| 2(8,1): | 2 | 2 | [C-,i1] | 0 | 3 | (*,*) | (4,*) | 1,2,1 |
| 3(9,1): | 2 | 2 | [C-,i1] | 1 | 2 | (*,*) | (4,*) | 1,2,1 |
| 4(10,0): | 3 | 3 | [rn,r1] | 4 | 4 | (*,*) | (2,3) | 1,2,1,2 |

(86)

To get a start, recall that the kgb orbit (that is $x$ in the pair $(x,y)$, or the first number in the pair $(i,j)$ of the `block` output) determines every piece of the $\Lambda$-data EXCEPT $\kappa$. Therefore, the data for representation #0 may be obtained from those for representation #4 in the large block by changing $\kappa$. So we have CSG $H_2$ as described in (63), and

$$\begin{aligned}
\lambda &= (2,-1) \\
\alpha_1 &= e_1 + e_2 \\
\alpha_2 &= -2e_2 \\
\Phi_R^+ &= \{2e_2\}
\end{aligned}$$

The choices for $\kappa$ are limited by the following conditions:

1. $\kappa$ must agree with $\lambda$ on the connected part of the compact torus:
$$\lambda + \theta\lambda = \kappa + \theta\kappa \tag{87}$$

2. Only its restriction to $H^\theta$ is important; i. e., $\kappa$ and $\kappa'$ determine the same character $\Lambda$ if and only if
$$\kappa - \kappa' \in (1-\theta)X^*(H). \tag{88}$$

In this example, this means that the first coordinate of $\kappa$ is forced to be 2, and the second coordinate must be an integer which is determined modulo 2 only. (This determines the character of the $\mathbb{Z}/2\mathbb{Z}$ factor in our Cartan.) Since $\kappa$ must be different from $\lambda$, we must have
$$\kappa = (2,0). \tag{89}$$

So $\Lambda$ is given by
$$\Lambda\left(e^{i\varphi}, r\right) = e^{2i\varphi}\,|r|^{-1}. \tag{90}$$

Similar arguments lead us to the $\Lambda$-data of #'s 1, 2, and 3, which we record in the table below.

To find the data for #4, we perform a Cayley transform through $\alpha_2 = 2e_1$ on #2. The new Cartan subgroup is the split one, $H_3$, and we get

$$\begin{aligned} \lambda &= (1,-2) \\ \kappa &= (1,1) \\ \alpha_1 &= -e_1 - e_2 \\ \alpha_2 &= 2e_1 \\ \Phi_R^+ &= \{e_1+e_2, -2e_1, 2e_2, -e_1+e_2\} \\ \Lambda(r_1,r_2) &= |r_1|\,|r_2|^{-2}\, sgn(r_1 r_2). \end{aligned}$$

This time we compute $\Gamma$ so that we can compare this better to the two principal series we already have:

$$\gamma = \lambda = (1,-2) \tag{91}$$

$$\xi = \kappa + \rho_R = (0,3) \tag{92}$$

$$\Gamma(r_1,r_2) = |r_1|\,|r_2|^{-2}\, sgn(r_2). \tag{93}$$

One can check that this is conjugate by the real Weyl group to

$$\Gamma'(r_1,r_2) = |r_1|^2\,|r_2|^1\, sgn(r_1). \tag{94}$$

This leaves the unique remaining principal series for the small block $Sp(4,\mathbb{R}) \times SO(5)$:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| `0(10,0):` | `3` | `3` | `[rn,rn]` | `0` | `0` | `(*,*)` | `(*,*)` | `1,2,1,2` | (95) |

$$\begin{aligned} \lambda &= (1,-2) \\ \kappa &= (0,0) \\ \alpha_1 &= -e_1 - e_2 \\ \alpha_2 &= 2e_1 \\ \Phi_R^+ &= \{e_1+e_2, -2e_1, 2e_2, -e_1+e_2\} \\ \Lambda(r_1,r_2) &= |r_1|\,|r_2|^{-2}. \end{aligned}$$

| # | $\theta$ | CSG | $\alpha_1$ | $\alpha_2$ | $\lambda$ | $\kappa$ |
|---|---|---|---|---|---|---|
| 0 $[SO(4,1)]$ | $s_{2e_2}$ | 2 | $e_1+e_2$ | $-2e_2$ | $(2,-1)$ | $(2,0)$ |
| 1 $[SO(4,1)]$ | $s_{2e_2}$ | 2 | $-e_1+e_2$ | $-2e_2$ | $(-2,-1)$ | $(-2,0)$ |
| 2 $[SO(4,1)]$ | $s_{2e_2}$ | 2 | $-e_1-e_2$ | $2e_1$ | $(1,-2)$ | $(1,1)$ |
| 3 $[SO(4,1)]$ | $s_{2e_2}$ | 2 | $e_1-e_2$ | $-2e_1$ | $(-1,-2)$ | $(-1,1)$ |
| 4 $[SO(4,1)]$ | inv | 3 | $-e_1-e_2$ | $2e_1$ | $(1,-2)$ | $(1,1)$ |
| 0 $[SO(5)]$ | inv | 3 | $-e_1-e_2$ | $2e_1$ | $(1,-2)$ | $(0,0)$ |

(96)

## 11 Other infinitesimal characters

Recall that the `block` output parametrizes translation families of representations, rather than individual ones. That is, we have a standard module and an irreducible for each line of the `block` output and infinitesimal character with the correct integrality and regularity properties. In the preceding exposition, we have fixed the infinitesimal character to be that of the trivial representation. Any other allowed infinitesimal character may be represented by a regular element $\psi \in \rho + X^*(H)$. Assuming that we have the $\Lambda$-data

$$(H(\mathbb{R}), \lambda, \kappa)$$

at infinitesimal character $\rho$ for a certain line of the `block` output, the corresponding $\Lambda$-data at infinitesimal character $\psi$ are then

$$(H(\mathbb{R}), \lambda', \kappa'),$$

where $\lambda'$ is the unique Weyl group conjugate of $\psi$ in the Weyl chamber of $\lambda$, and

$$\kappa' = \lambda' + (\kappa - \lambda).$$

Notice that this says that the quantity

$$\eta = \kappa - \lambda$$

is an invariant of each representation that is independent of the infinitesimal character.

For example, the representation at infinitesimal character $(5, 3)$ assigned to #4 of the large block for $Sp(4, \mathbb{R})$ has $\Lambda$-data

$$\begin{aligned} H(\mathbb{R}) &= H_2, \\ \lambda &= (5, -3) = \kappa, \end{aligned}$$

the one assigned to #11 has

$$\begin{aligned} H(\mathbb{R}) &= H_3, \\ \lambda &= (5, 3), \\ \kappa &= (6, 2). \end{aligned}$$

## 12 The Levi subgroup $M$

Given a Cartan subgroup $H(\mathbb{R})$ and a character $\Gamma$, the construction of the corresponding standard module may be done by real parabolic induction, as outlined in Section 1. The type of the corresponding root system is given by the type of the system of imaginary roots with respect to $H(\mathbb{R})$ (this can be found in the `cartan` output). We can determine the real form of the derived group $M_d$ of $M$ by looking at the nature of the imaginary roots. By cross actions, we can arrange for the roots that are simple for the subsystem of imaginary roots to be simple for the whole root system. In the `block` output lines corresponding to our CSG $H(\mathbb{R})$, find the ones with the number of simple imaginary roots equal to the rank of the imaginary root system. The nature of these roots indicates the real form of $M_d$. If all of them are compact, then so is $M_d$. If not, look at the lines with all imaginary simple roots except one being compact (on each simple factor). They correspond to the discrete series attached to the Borel de Siebenthal chamber, and the node in the Dynkin diagram corresponding to the noncompact root determines the real form.

The Levi subgroup $M$ is then a group (real points of a connected complex algebraic group) containing $H(\mathbb{R})$ as the fundamental Cartan, and containing $M_d$. This information does not necessarily determine $M$ uniquely, but in general, there are not many possibilities. Here are some more facts which help in determining $M$:

- If $G(\mathbb{R})$ is simply connected, then so is $M$ (here we mean that they are real points of a simply connected algebraic group).
- If $G(\mathbb{R})$ is quasisplit, then so is $M$.

For example, take $H(\mathbb{R}) = H_1 \simeq \mathbb{C}^\times$ of our $Sp(4,\mathbb{R})$ example. The `cartan` output tells us that the imaginary root system is of type $A_1$. Since $Sp(4,\mathbb{R})$ is simply connected and split, $M_d = SL(2,\mathbb{R})$. So we are looking for a split group of type $A_1$ with fundamental Cartan subgroup $\mathbb{C}^\times$; this is $GL(2,\mathbb{R})$. In the notation of (2), $M' = SL(2,\mathbb{R})^\pm$, and $A = \mathbb{R}^\times_+$. For representation #6, for example, with $\Lambda$ and $\Gamma$ given by (37) and (39), the discrete series $\sigma$ of $M'$ (see(3)) has Harish-Chandra parameter 3 and lowest $O(2)$-type (4) (this is the discrete series whose restriction to $SL(2,\mathbb{R})$ is the sum of the holomorphic and antiholomorphic discrete series at infinitesimal character 3, with lowest $SO(2)$-types 4 and $-4$). The character $\nu$ of $A$ is given by the identity character.

If $H(\mathbb{R}) = H_2 \simeq S^1 \times \mathbb{R}^\times$, we have again that the imaginary root system is of type $A_1$, but then $M = SL(2,\mathbb{R}) \times \mathbb{R}^\times$. Now $M' = SL(2,\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}$. For representation #4, for example, the formula (69) for $\Gamma$ says that $\sigma$ is the discrete series of $SL(2,\mathbb{R})$ with lowest $SO(2)$-type (3), tensored with the sign character of $\mathbb{Z}/2\mathbb{Z}$, and $\nu$ is the inverse character $r \longmapsto r^{-1}$.

For a more interesting, and less familiar, example, consider the non-quasisplit noncompact real form of $E_6$ in the equal rank inner class (maximal compact subgroup $SO(10)\times U(1)$), simply connected. This group has a Cartan subgroup $H_1 \simeq (S^1)^4 \times \mathbb{C}^\times$:

```
Cartan #1:
```

```
split:  0; compact:  4; complex:  1
canonical twisted involution:  2,4,3,1,5,4,2,3,4,5,6,5,4,2,3,1,4,3,5,4,2
twisted involution orbit size:  36; fiber rank:  4; #X_r:  576
imaginary root system:  A5
real root system:  A1
complex factor is empty
```

The imaginary root system is of type $A_5$, so $M_d$ is a real form of $SL(6,\mathbb{C})$. To see which, we look in the `block` output (there is only one block for this real form) to find a line attached to Cartan #1, with five imaginary simple roots, at least four of which are compact:

```
472(472,359): 11 1 [i1,C-,ic,ic,ic,ic]...
```

Keeping in mind the `atlas` (Bourbaki) numbering of the roots of $E_6$

$$\begin{array}{ccc} & 2 & \\ & | & \\ 1-3- & 4-5-6 & \end{array}$$

the imaginary roots give us the Dynkin diagram for $A_5$ with the first root noncompact. This corresponds to the real form $SU(1,5)$ or $SU(5,1)$. Our Levi subgroup $M$ must have derived group $SU(5,1)$ and fundamental Cartan $H_1$; this is the group

$$\left(SU(5,1)\times\mathbb{R}^\times\right)/\left\langle(-I,-1)\right\rangle.$$

This group is a subgroup of $GU(5,1)$; it has center $\langle\mathbb{R}^\times,\varsigma\rangle$, where $\zeta$ is a cube root of unity.

# 13 More examples

## 13.1 $SL(2,\mathbb{R})$ and $PGL(2,\mathbb{R})\simeq SO(2,1)$

We consider the basic example of type $A_1$.

### 13.1.1 SL(2,$\mathbb{R}$)

We start with the split, simply connected group $SL(2,\mathbb{R})$. Write

$$H=\mathbb{C}^\times=\left\{re^{i\varphi}\right\}, \tag{97}$$

and

$$\Psi = \{\pm 2e\}\,,$$

with the root reflection acting by inversion. The group has two conjugacy classes of Cartan subgroups, $H_0 = S^1$ and $H_1 = \mathbb{R}^\times$. The quasisplit form of the dual group is $PGL(2,\mathbb{R}) \simeq SO(2,1)$. Here is the `kgb` output for this group:

```
0:  0  0  [n]  0  1
1:  1  1  [r]  1  *  1
```

There is only one choice of basepoints; both orbits are basepoint orbits, orbit 0 being the fundamental one. So we have the large block

```
0(0,1):  0  0  [i1]  1  (2,*)      B
1(1,1)   0  0  [i1]  0  (2,*)      B
2(2,0):  1  1  [r1]  2  (0,1)  1   B
```

Lines 0 and 1 are indistinguishable; they are associated with the compact Cartan $H_0$ and hence the two discrete series at infinitesimal character $\rho$. We must choose an assignment. We choose #0 to be the representation with HC parameter

$$\lambda = (1) = \kappa,$$

so the simple root

$$\begin{aligned} \alpha &= 2e. \\ \rho_i &= (1), \\ \rho_{i,c} &= 0, \\ \gamma &= (2) = \xi \text{ (the lowest } K\text{-type)} \end{aligned}$$

We get the other discrete series by cross action through $\alpha = 2e$:

$$\begin{aligned} \lambda &= (-1) = \kappa, \\ \alpha &= -2e, \\ \gamma &= (-2) = \xi. \end{aligned}$$

We obtain the parameters for #2 from #0 by Cayley transform through $\alpha$:

$$\begin{aligned} \lambda &= (1) = \kappa, \\ \alpha &= 2e, \\ \rho_R &= (-1), \\ \gamma &= (1), \\ \xi &= (0). \end{aligned}$$

We have for $r \in H_1 = \mathbb{R}^\times$,

$$\begin{aligned}\Lambda(r) &= r, \\ \Gamma(r) &= |r|,\end{aligned}$$

which is, of course, the inducing data for the trivial representation/spherical principal series.

Since $SL(2,\mathbb{R})$ is simply connected, the small block (dual group $SO(3)$) is also attached to infinitesimal character $\rho$:

```
0(2,0):  1  1  [rn]  0  (*,*)  1
```

This must be the nonspherical principal series

$$\begin{aligned}\lambda &= (1), \\ \kappa &= (2) \\ \alpha &= 2e, \\ \rho_R &= (-1), \\ \gamma &= (1), \\ \xi &= (1), \\ \Lambda(r) &= |r|, \\ \Gamma(r) &= r.\end{aligned}$$

So we get a table for $SL(2,\mathbb{R})$:

| # | $\theta$ | CSG | $\alpha$ | $\lambda$ | $\kappa$ |
|---|---|---|---|---|---|
| 0 $[SO(2,1)]$ | 1 | 0 | $2e$ | $(1)$ | $(1)$ |
| 1 $[SO(2,1)]$ | 1 | 0 | $-2e$ | $(-1)$ | $(-1)$ |
| 2 $[SO(2,1)]$ | inv | 1 | $2e$ | $(1)$ | $(1)$ |
| 0 $[SO(3)]$ | inv | 1 | $2e$ | $(1)$ | $(2)$ |

(98)

### 13.1.2 SO(2,1)

For the adjoint group, $G(\mathbb{R}) = SO(2,1)$, we need to look at the `kgb` output of the simply connected split form $SL(2,\mathbb{R})$:

```
0:  0  0  [n]  1  2
1:  0  0  [n]  0  2
2:  1  1  [r]  2  *  1
```

This time, we have a choice to make for our fundamental basepoint since lines 0 and 1 are indistinguishable. We choose 0 as our fundamental basepoint, then 2 will be the basepoint for the split Cartan. So we get the large block for $SO(2,1)$:

```
0(0,2):  0  0  [i2]  0  (1,2)     B
1(1,0)   1  1  [r2]  2  (0,*)  1  B
2(1,1)   1  1  [r2]  1  (0,*)  1
```

We can choose our complex Cartan and the two real forms as for $SL(2,\mathbb{R})$; however,

$$\Psi = \{\pm e\},$$

and

$$\rho = \begin{pmatrix} \frac{1}{2} \end{pmatrix}$$

(up to Weyl group conjugation) does not belong to $X^*(H)$, so the $\rho$-cover of $H(\mathbb{R})$ is not trivial this time.

Representation #0 is the unique discrete series, with

$$\begin{aligned} \lambda &= \begin{pmatrix} \frac{1}{2} \end{pmatrix} = \kappa, \\ \alpha &= e, \\ \rho_i &= \begin{pmatrix} \frac{1}{2} \end{pmatrix}, \\ \rho_{i,c} &= 0, \\ \gamma &= (1) = \xi. \end{aligned}$$

Notice that lines 1 and 2 are indistinguishable; they correspond to the two one-dimensional representations of $SO(2,1)$. Choosing `kgb` orbit 0 as our fundamental basepoint above amounted to choosing representation #1 to be the trivial representation. We obtain the parameters by Cayley transform from #0:

$$\begin{aligned} \lambda &= \begin{pmatrix} \frac{1}{2} \end{pmatrix} = \kappa, \\ \alpha &= e, \\ \rho_R &= \begin{pmatrix} -\frac{1}{2} \end{pmatrix}, \\ \gamma &= \begin{pmatrix} \frac{1}{2} \end{pmatrix}, \\ \xi &= (0), \\ \Gamma(r) &= |r|^{\frac{1}{2}}. \end{aligned}$$

We obtain the data for #2 by cross action through the real root $\alpha$:

$$\begin{aligned}
\lambda &= \left(\frac{1}{2}\right), \\
\kappa &= \left(\frac{3}{2}\right), \\
\alpha &= e, \\
\rho_R &= \left(-\frac{1}{2}\right), \\
\gamma &= \left(\frac{1}{2}\right), \\
\xi &= (1), \\
\Gamma(r) &= |r|^{\frac{1}{2}}\, sgn(r).
\end{aligned}$$

This is the sign representation of $SO(2,1)$. We have the table

| # | $\theta$ | CSG | $\alpha$ | $\lambda$ | $\kappa$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | $e$ | $(\frac{1}{2})$ | $(\frac{1}{2})$ |
| 1 | inv | 1 | $e$ | $(\frac{1}{2})$ | $(\frac{1}{2})$ |
| 2 | inv | 1 | $e$ | $(\frac{1}{2})$ | $(\frac{3}{2})$ |

(99)

Now the question arises whether these are all the representations at this infinitesimal character. To find the blocks attached to the translation family of $\rho$, we check which strong real forms on the dual side belong to the same "real form class" as the quasisplit one, using the command **strongreal**. For $G^\vee = SL(2,\mathbb{C})$, this gives

```
empty: type
Lie type: A1 sc s
main: strongreal
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
cartan class (one of 0,1): 0
Name an output file (return for stout, ? to abandon):
there are 2 real form classes:
class #0:
real form #1: [0,1] (2)
class #1:
real form #0: [0] (1)
```

```
real form #0:  [1] (1)
```

The real form #1 is the split one; it is in class #0, so class #0 gives the blocks attached to infinitesimal character $\rho$. Since it contains only one strong real form, the large block is the only one. There are two more blocks, for dual groups $SU(2,0)$ and $SU(0,2)$; they give the representations of $SO(2,1)$ with infinitesimal character (1).(there are two principal series). `atlas` outputs only one of these two blocks since they look the same.

## 13.2 $GL(2,\mathbb{R})$

As an example of a reductive (non-semisimple) group, we look at $GL(2,\mathbb{R})$. The quasisplit form of the dual group is $U(1,1)$, which has `kgb` output

```
0:  0  0  [n]  1  2
1:  0  0  [n]  0  2
2:  1  1  [r]  2  *  1
```

Lines 0 and 1 are indistinguishable; we choose 0 as our fundamental basepoint. This gives for our `block` output:

```
0(0,2):  0  0  [i2]  0  (1,2)     B
1(1,0):  1  1  [r2]  2  (0,*)  1  B
2(1,1):  1  1  [r2]  1  (0,*)  1
```

Our group $GL(2,\mathbb{R})$ has rank two and two Cartan subgroups, one complex (#0), and one split (#1). Write

$$\begin{aligned} H &= \left(\mathbb{C}^{\times}\right)^{2}, \\ \Psi &= \left\{\pm\left(e_1 - e_2\right)\right\}, \end{aligned}$$

with the root reflection acting by transposition of the two entries. If the split Cartan $H_1$ corresponds to $\theta_1$ being inversion, then $\theta_0$ must be given by

$$\theta_0(z_1, z_2) = (z_2^{-1}, z_1^{-1}),$$

so that

$$H_0 = \left\{h_{r,\varphi} = \left(re^{i\varphi}, re^{-i\varphi}\right)\right\} \simeq \mathbb{C}^{\times}.$$

The unique representation attached to the fundamental Cartan has parameters

$$\begin{aligned} \lambda &= \left(\frac{1}{2}, -\frac{1}{2}\right) = \kappa, \\ \gamma &= (1,-1) = \xi, \end{aligned}$$

and we get the parameters of the two principal series by Cayley transform and cross action. By our choice of basepoints, #1 is the trivial representation, and #2 the sign of the determinant. We record the Λ-data in the table. Notice that for #1,

$$\Gamma(r_1, r_2) = |r_1|^{\frac{1}{2}} |r_2|^{-\frac{1}{2}},$$

and for #2,

$$\Gamma(r_1, r_2) = |r_1|^{\frac{1}{2}} |r_2|^{-\frac{1}{2}} sgn(r_1 r_2).$$

| # | $\theta$ | CSG | $\alpha$ | $\lambda$ | $\kappa$ |
|---|---|---|---|---|---|
| 0 | inv$\circ s_\alpha$ | 0 | $e_1 - e_2$ | $\left(\frac{1}{2}, -\frac{1}{2}\right)$ | $\left(\frac{1}{2}, -\frac{1}{2}\right)$ |
| 1 | inv | 1 | $e_1 - e_2$ | $\left(\frac{1}{2}, -\frac{1}{2}\right)$ | $\left(\frac{1}{2}, -\frac{1}{2}\right)$ |
| 2 | inv | 1 | $e_1 - e_2$ | $\left(\frac{1}{2}, -\frac{1}{2}\right)$ | $\left(\frac{3}{2}, -\frac{3}{2}\right)$ |

(100)

## 13.3 $SL(3, \mathbb{R})$

In this section, we finish the example started in Section 5.2. The quasisplit form of the dual group is $PSU(2,1)$, which has `kgb` output

```
0:  0  0  [n,n]  1  2  4  3
1:  0  0  [n,c]  0  1  4  *
2:  0  0  [c,n]  2  0  *  3
3:  1  1  [C,r]  5  3  *  *  2
4:  1  1  [r,C]  4  5  *  *  1
5:  2  1  [C,C]  3  4  *  *  1,2,1
```

The basepoints are 0, 3, 4, and 5, which we indicate in the `block` output; we are adding the small block $(SL(3,\mathbb{R}) \times PSU(3))$ as well:

```
0  (0,5):  0  0  [C+,C+]  2  1  (*,*)  (*,*)         B
1  (1,4)   1  0  [i2,C-]  1  0  (3,4)  (*,*)  2,1    B
2  (2,3)   1  0  [C-,i2]  0  2  (*,*)  (3,5)  1,2    B
3  (3,0)   2  1  [r2,r2]  4  5  (1,*)  (2,*)  1,2,1  B
4  (3,1)   2  1  [r2,rn]  3  4  (1,*)  (*,*)  1,2,1
5  (3,2)   2  1  [rn,r2]  5  3  (*,*)  (2,*)  1,2,1

0  (3,1)   2  1  [rn,rn]  0  0  (*,*)  (*,*)  1,2,1
```

(101)

We have the data for #0, and we obtain the data for the other two representations attached to the fundamental Cartan by complex cross actions as explained in Section 7

The remaining representations are the four principal series representations of $SL(3,\mathbb{R})$, attached to the split Cartan

$$H_1 = \left\{(r_1, r_2, r_3) \in \left(\mathbb{R}^\times\right)^3 : r_1 r_2 r_3 = 1\right\}.$$

The basepoint (hence trivial) representation is #3, we obtain the parameters by Cayley transform from #0 through $\alpha_2 = e_2 - e_3$:

$$\begin{aligned} \lambda &= (1, 0, -1) = \kappa \\ \Lambda(r_1, r_2, r_3) &= r_1 r_3^{-1} \\ \Gamma(r_1, r_2, r_3) &= |r_1|\,|r_3|^{-1} \end{aligned}$$

The data for the remaining representations in the large block are obtained by real cross actions, which change the $\kappa$ parameters only.

To determine the parameter $\kappa$ for the representation in the small block, we write the three known characters $\Lambda$ so that we can easily compare them, and check which one is missing:

$$\begin{aligned} \Lambda_3(r_1, r_2, r_3) &= |r_1|\,|r_3|^{-1}\, sgn(r_1 r_3) \\ \Lambda_4(r_1, r_2, r_3) &= |r_1|\,|r_3|^{-1}\, sgn(r_2 r_3) = |r_1|\,|r_3|^{-1}\, sgn(r_1) \\ \Lambda_5(r_1, r_2, r_3) &= |r_1|\,|r_3|^{-1}\, sgn(r_1 r_2) = |r_1|\,|r_3|^{-1}\, sgn(r_3) \end{aligned}$$

So we must have for the last character

$$\begin{aligned} \Lambda(r_1, r_2, r_3) &= |r_1|\,|r_3|^{-1}, \\ \kappa &= (0, 0, 0). \end{aligned}$$

We record our results in a table.

| # | $\theta$ | CSG | $\alpha_1$ | $\alpha_2$ | $\lambda$ | $\kappa$ |
|---|---|---|---|---|---|---|
| 0 | $inv \circ s_{e_1-e_3}$ | 0 | $e_1 - e_2$ | $e_2 - e_3$ | $(1,0,-1)$ | $(1,0,-1)$ |
| 1 | $inv \circ s_{e_1-e_3}$ | 0 | $e_1 - e_3$ | $-e_2 + e_3$ | $(1,-1,0)$ | $(1,-1,0)$ |
| 2 | $inv \circ s_{e_1-e_3}$ | 0 | $-e_1 + e_2$ | $e_1 - e_3$ | $(0,1,-1)$ | $(0,1,-1)$ |
| 3 | inv | 1 | $e_1 - e_2$ | $e_2 - e_3$ | $(1,0,-1)$ | $(1,0,-1)$ |
| 4 | inv | 1 | $e_1 - e_2$ | $e_2 - e_3$ | $(1,0,-1)$ | $(2,-1,-1)$ |
| 5 | inv | 1 | $e_1 - e_2$ | $e_2 - e_3$ | $(1,0,-1)$ | $(1,1,-2)$ |
| $0[PSU(3)]$ | inv | 1 | $e_1 - e_2$ | $e_2 - e_3$ | $(1,0,-1)$ | $(0,0,0)$ |

(102)

### 13.4 Split $G_2$

The group $G_2$ is both simply connected and adjoint, and dual to itself. Below is the `block` output for the split form of $G_2$; we have added the basepoints, for which there is a unique choice (as happens whenever the dual group is adjoint):

```
0(0,9):   0  0  [i1,i1]  1   2   (3,*)   (4,*)                 B
1(1,9):   0  0  [i1,ic]  0   1   (3,*)   (*,*)                 B
2(2,9):   0  0  [ic,i1]  2   0   (*,*)   (4,*)                 B
3(3,7):   1  1  [r1,C+]  3   6   (0,1)   (*,*)   1             B
4(4,8):   1  2  [C+,r1]  5   4   (*,*)   (0,2)   2             B
5(5,5):   2  2  [C-,C+]  4   8   (*,*)   (*,*)   1,2,1         B
6(6,6):   2  1  [C+,C-]  7   3   (*,*)   (*,*)   2,1,2         B
7(7,3):   3  1  [C-,i2]  6   7   (*,*)   (9,11)  1,2,1,2,1     B
8(8,4):   3  2  [i2,C-]  8   5   (9,10)  (*,*)   2,1,2,1,2     B
9(9,0):   4  3  [r2,r2]  10  11  (8,*)   (7,*)   2,1,2,1,2,1   B
10(9,1):  4  3  [r2,rn]  9   10  (8,*)   (*,*)   2,1,2,1,2,1
11(9,2):  4  3  [rn,r2]  11  9   (*,*)   (7,*)   2,1,2,1,2,1
```
(103)

There are four conjugacy classes of Cartan subgroups; one compact (#0), one split (#3), and two complex, distinguished by whether the imaginary roots are short (#2) or long (#1).

We choose our root system as follows. The complex Cartan $H$ is

$$H = \left\{(z_1, z_2, z_3) \in \left(\mathbb{C}^\times\right)^3 : z_1 z_2 z_3 = 1\right\} \simeq \left(\mathbb{C}^\times\right)^2, \tag{104}$$

and our root system

$$\Psi = \left\{ \begin{array}{c} \pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3), \\ \pm(e_1 + e_2 - 2e_3), \pm(e_1 - 2e_2 + e_3), \pm(2e_1 - e_2 - e_3) \end{array} \right\} \tag{105}$$

The root reflections act as follows:

$$s_{e_1 - e_2}(z_1, z_2, z_3) = (z_2, z_1, z_3),$$

and analogously for the other short roots; and

$$s_{2e_1 - e_2 - e_3}(z_1, z_2, z_3) = (z_2^{-1}, z_1^{-2}, z_3^{-1}),$$

and analogously for the other long roots. Consequently, the Weyl group acts by permuations and sign change/inversion of all three coordinates at once, and is isomorphic to $S_3 \times \mathbb{Z}/2\mathbb{Z}$.

`atlas` chooses $\alpha_1$ to be the short simple root, and $\alpha_2$ to be the long one.

The fundamental Cartan is compact, corresponding to $\theta = 1$:

$$H_0 = \left\{(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}) : \varphi_1 + \varphi_2 + \varphi_3 \in 2\mathbb{Z}\right\}. \tag{106}$$

The standard choice of (positive) compact roots (according to, e. g., [12]), is

$$\Psi_{i,c}^+ = \{(e_2 - e_3), (2e_1 - e_2 - e_3)\} \subseteq \Psi_{i,c} = \{\pm(e_2 - e_3), \pm(2e_1 - e_2 - e_3)\}. \tag{107}$$

A weight

$$(a, b, c)$$

is dominant with respect to $\Psi_{i,c}^+$ if and only if

$$a \geq 0 \text{ and } b \geq c.$$

In these coordinates, $\rho$ is any Weyl group conjugate of

$$(3, -1, -2).$$

The large discrete series (#0 in the block output) has Harish-Chandra parameter

$$\lambda = (2, 1, -3) = \kappa,$$

with

$$\begin{aligned} \alpha_1 &= e_1 - e_2, \\ \alpha_2 &= -e_1 + 2e_2 - e_3. \end{aligned}$$

So we have

$$\Lambda(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}) = e^{i(2\varphi_1 + \varphi_2 - 3\varphi_3)}.$$

$$\begin{aligned} \Psi_{i,n}^+ &= \{e_1 - e_2, -e_1 + 2e_2 - e_3, e_1 - e_3, e_1 + e_2 - 2e_3\}, \\ \rho_i &= (3, -1, -2), \\ 2\rho_{i,c} &= (2, 0, -2), \\ \gamma &= (4, -2, -2) = \xi. \end{aligned}$$

We obtain the paramters for the other two discrete series, #1 and #2, by cross action through $\alpha_1 = e_1 - e_2$ and $\alpha_2 = -e_1 + 2e_2 - e_3$, respectively. So we have for #1:

$$\lambda = (1, 2, -3) = \kappa,$$

$$\begin{aligned}\alpha_1 &= -e_1 + e_2,\\ \alpha_2 &= 2e_1 - e_2 - e_3.\end{aligned}$$

and for #2:

$$\lambda = (3, -1, -2) = \kappa,$$

$$\begin{aligned}\alpha_1 &= e_2 - e_3,\\ \alpha_2 &= e_1 - 2e_2 + e_3.\end{aligned}$$

To get to Cartan #1, we perform a Cayley transform on representation #0, through $\alpha_1 = e_1 - e_2$. Then $\theta = \theta_1 = s_{e_1 - e_2}$, and

$$H_1 = \left\{h_{r,\varphi} = (re^{i\varphi}, r^{-1}e^{i\varphi}, e^{-2i\varphi})\right\} \simeq \mathbb{C}^\times.$$

We have

$$\begin{aligned}\lambda &= (2, 1, -3) = \kappa,\\ \alpha_1 &= e_1 - e_2,\\ \alpha_2 &= -e_1 + 2e_2 - e_3.\end{aligned}$$

Now $\alpha_1$ is real, $e_1 + e_2 - 2e_3$ is noncompact imaginary, and the other positive roots are complex, so we get

$$\begin{aligned}\Phi_R^+ &= \{-e_1 + e_2\},\\ \rho_R &= \left(-\frac{1}{2}, \frac{1}{2}, 0\right),\\ \rho_i &= \left(\frac{1}{2}, \frac{1}{2}, -1\right),\\ 2\rho_{i,c} &= 0.\end{aligned}$$

We have

$$\theta_1(e_1 - e_3) = e_2 - e_3, \text{ and } \theta_1(2e_1 - e_2 - e_3) = -e_1 + 2e_2 - e_3,$$

so we can take

$$\rho_{cx}^\circ = (2, -2, 0).$$

So we compute

$$\begin{aligned}\gamma &= \left(\frac{5}{2}, \frac{3}{2}, -4\right),\\ \xi &= (4, 0, -4),\end{aligned}$$

giving us

$$\begin{aligned}\Lambda(h_{r,\varphi}) &= re^{9i\varphi},\\ \Gamma(h_{r,\varphi}) &= re^{12i\varphi}.\end{aligned}$$

Notice that $\rho_{cx}^{\circ}$ and $\rho_R$ both belong to $(1-\theta_1)X^*(H)$, so we could actually take $\xi = \gamma$ instead (as expected, since $H_1$ is connected).

We get the parameters for #6 from those for #3 by cross action through $\alpha_2 = -e_1 + 2e_2 - e_3$, and those for #7 from those for #6 by cross action through $\alpha_1 = e_2 - e_3$. We leave the details to the reader, and record the $\Lambda$-data in the table (108) below.

A Cayley transform from #0 through the long root $\alpha_2 = -e_1 + 2e_2 - e_3$ gets us to representation #4 , which is attached to CSG #2. We have $\theta_2 = s_{-e_1+2e_2-e_3}$, and

$$H_2 = H(\mathbb{R}) = \left\{h_{r,\varphi}\left(re^{i\varphi}, r^{-2}, re^{-i\varphi}\right)\right\} \simeq \mathbb{C}^{\times}.$$

Since this is connected, we can take $\lambda = \kappa$ and $\gamma = \xi$, and don't need to compute $\rho_{cx}^{\circ}$ or $\rho_R$. For #4, the data are

$$\begin{aligned}\lambda &= (2,1,-3) = \kappa,\\ \alpha_1 &= e_1 - e_2,\\ \alpha_2 &= -e_1 + 2e_2 - e_3,\\ \Phi_R^+ &= \{e_1 - 2e_2 + e_3\},\\ \rho_i &= \left(\frac{1}{2}, 0, -\frac{1}{2}\right),\\ \rho_{i,c} &= 0,\\ \gamma &= \left(\frac{5}{2}, 1, -\frac{7}{2}\right),\\ \Lambda(h_{r,\varphi}) &= r^{-3}e^{5i\varphi},\\ \Gamma(h_{r,\varphi}) &= r^{-3}e^{6i\varphi}.\end{aligned}$$

We leave #5 and #8 to the reader, and just record the $\Lambda$-data below.

For the principal series, we start by doing a Cayley transform to the parameters of #7 through $\alpha_2 = e_1 + e_2 - 2e_3$. The Cartan subgroup is split

$$H_3 = \left\{(r_1, r_2, r_3) \in \left(\mathbb{R}^{\times}\right)^3 : r_1 r_2 r_3 = 1\right\} \simeq \left(\mathbb{R}^{\times}\right)^2.$$

The basepoint representation (with $\lambda = \kappa$) is #9 (this must be the spherical one); it has

$$\begin{aligned}\lambda &= (3,-2,-1)=\kappa,\\ \alpha_1 &= -e_2+e_3,\\ \alpha_2 &= e_1+e_2-2e_3,\\ \rho_R &= -\lambda.\end{aligned}$$

All roots are real, so

$$\begin{aligned}\gamma &= (3,-2,-1),\\ \xi &= (0,0,0),\\ \Lambda(r_1,r_2,r_3) &= r_1^3 r_2^{-2} r_3^{-1},\\ \Gamma(r_1,r_2,r_3) &= |r_1|^3 |r_2|^{-2} |r_3|^{-1}.\end{aligned}$$

We get #10 and #11 from #9 by cross actions through $\alpha_1 = -e_2+e_3$ and $\alpha_2 = e_1+e_2-2e_3$, respectively. This gives

$$\begin{aligned}\Lambda(r_1,r_2,r_3) &= r_1^3 r_2^{-2} r_3^{-1} sgn(r_2 r_3) \text{ for \#10, and}\\ \Lambda(r_1,r_2,r_3) &= r_1^3 r_2^{-2} r_3^{-1} sgn(r_1 r_2) \text{ for \#11.}\end{aligned}$$

By calculation of the parity condition, or by symmetry considerations, we get that the principal series in the small block must correspond to

$$\begin{aligned}\kappa &= (4,-2,-2),\\ \Lambda(r_1,r_2,r_3) &= r_1^3 r_2^{-2} r_3^{-1} sgn(r_1 r_3)\end{aligned}$$

| # | $\theta$ | CSG | $\alpha_1$ | $\alpha_2$ | $\lambda$ | $\kappa$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | $e_1-e_2$ | $-e_1+2e_2-e_3$ | $(2,1,-3)$ | $(2,1,-3)$ |
| 1 | 1 | 0 | $-e_1+e_2$ | $2e_1-e_2-e_3$ | $(1,2,-3)$ | $(1,2,-3)$ |
| 2 | 1 | 0 | $e_2-e_3$ | $e_1-2e_2+e_3$ | $(3,-1,-2)$ | $(3,-1,-2)$ |
| 3 | $s_{e_1-e_2}$ | 1 | $e_1-e_2$ | $-e_1+2e_2-e_3$ | $(2,1,-3)$ | $(2,1,-3)$ |
| 4 | $s_{-e_1+2e_2-e_3}$ | 2 | $e_1-e_2$ | $-e_1+2e_2-e_3$ | $(2,1,-3)$ | $(2,1,-3)$ |
| 5 | $s_{-e_1+2e_2-e_3}$ | 2 | $-e_1+e_2$ | $2e_1-e_2-e_3$ | $(1,2,-3)$ | $(1,2,-3)$ |
| 6 | $s_{e_1-e_2}$ | 1 | $e_2-e_3$ | $e_1-2e_2+e_3$ | $(3,-1,-2)$ | $(3,-1,-2)$ |
| 7 | $s_{e_1-e_2}$ | 1 | $-e_2+e_3$ | $e_1+e_2-2e_3$ | $(3,-2,-1)$ | $(3,-2,-1)$ |
| 8 | $s_{-e_1+2e_2-e_3}$ | 2 | $e_1-e_3$ | $-2e_1+e_2+e_3$ | $(-1,3,-2)$ | $(-1,3,-2)$ |
| 9 | inv | 3 | $-e_2+e_3$ | $e_1+e_2-2e_3$ | $(3,-2,-1)$ | $(3,-2,-1)$ |
| 10 | inv | 3 | $-e_2+e_3$ | $e_1+e_2-2e_3$ | $(3,-2,-1)$ | $(3,-3,0)$ |
| 11 | inv | 3 | $-e_2+e_3$ | $e_1+e_2-2e_3$ | $(3,-2,-1)$ | $(4,-1,-3)$ |
| C | inv | 3 | $-e_2+e_3$ | $e_1+e_2-2e_3$ | $(3,-2,-1)$ | $(4,-2,-2)$ |

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### 13.5 $SO(4,1)$

As an example of a non-quasisplit real form, we look at the example $SO(4,1)$. The (quasi)split form of the dual group is $Sp(4,\mathbb{R})$, with `kgb` output

```
0:   0  0  [n,n]  1   2   6   4
1:   0  0  [n,n]  0   3   6   5
2:   0  0  [c,n]  2   0   *   4
3:   0  0  [c,n]  3   1   *   5
4:   1  2  [C,r]  8   4   *   *   2
5:   1  2  [C,r]  9   5   *   *   2
6:   1  1  [r,C]  6   7   *   *   1
7:   2  1  [n,C]  7   6   10  *   2,1,2
8:   2  2  [C,n]  4   9   *   10  1,2,1
9:   2  2  [C,n]  5   8   *   10  1,2,1
10:  3  3  [r,r]  10  10  *   *   1,2,1,2
```

If we choose 0 as our fundamental bsepoint, we get basepoints 0,4,6,7,8,10. Here is the (large) block for $SO(4,1)$ with this information added:

```
0(0,10):  0  0  [ic,i2]  0  0  (*,*)  (1,2)         B
1(1,8):   1  1  [C+,r2]  3  2  (*,*)  (0,*)  2      B
2(1,9):   1  1  [C+,r2]  4  1  (*,*)  (0,*)  2
3(2,4):   2  1  [C-,ic]  1  3  (*,*)  (*,*)  1,2,1  B
4(2,5):   2  1  [C-,ic]  2  4  (*,*)  (*,*)  1,2,1
```

Write

$$\begin{aligned} H &= \left(\mathbb{C}^{\times}\right)^{2}, \\ \Psi &= \{\pm(e_1 - e_2), \pm(e_1 + e_2), \pm e_1, \pm e_2\}. \end{aligned}$$

`atlas` chooses the long simple root to be $\alpha_1$, the short one to be $\alpha_2$. There are two conjugacy classes of real CSG's, the fundamental one being compact, corresponding to $\theta = 1$. The compact roots are the long roots. There is one discrete series (#0):

$$\begin{aligned} \lambda &= \left(\frac{3}{2}, \frac{1}{2}\right) = \kappa \\ \alpha_1 &= e_1 - e_2 \\ \alpha_2 &= e_2 \\ \rho_i &= \left(\frac{3}{2}, \frac{1}{2}\right), \\ 2\rho_{i,c} &= (2,0), \\ \gamma &= (1,1) = \xi. \end{aligned}$$

We perform a Cayley transform through $\alpha_2 = e_2$ to get the parameters for #1: The Cartan subgroup is associated to $\theta_1 = s_{e_2}$,

$$H_1 = S^1 \times \mathbb{R}^\times.$$

The data for #1 are

$$\begin{aligned}
\lambda &= \left(\frac{3}{2}, \frac{1}{2}\right) = \kappa, \\
\alpha_1 &= e_1 - e_2, \\
\alpha_2 &= e_2, \\
\rho_R &= \left(0, -\frac{1}{2}\right), \\
\rho_i &= \left(\frac{1}{2}, 0\right), \\
\rho_{i,c} &= 0, \\
\rho^\circ_{cx} &= (0,1), \\
\gamma &= \left(2, \frac{1}{2}\right), \\
\xi &= (2,1), \\
\Gamma(e^{i\varphi}, r) &= e^{2i\varphi} |r|^{\frac{1}{2}} \, sgn(r).
\end{aligned}$$

We get #2 by cross action through the real root $\alpha_2 = e_2$ (only $\kappa$ and $\xi$ change), and the remaining two representations by more cross actions. Representation #3 is the trivial representation.

| # | $\theta$ | CSG | $\alpha_1$ | $\alpha_2$ | $\lambda$ | $\kappa$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | $e_1 - e_2$ | $e_2$ | $\left(\frac{3}{2}, \frac{1}{2}\right)$ | $\left(\frac{3}{2}, \frac{1}{2}\right)$ |
| 1 | $s_{e_2}$ | 1 | $e_1 - e_2$ | $e_2$ | $\left(\frac{3}{2}, \frac{1}{2}\right)$ | $\left(\frac{3}{2}, \frac{1}{2}\right)$ |
| 2 | $s_{e_2}$ | 1 | $e_1 - e_2$ | $e_2$ | $\left(\frac{3}{2}, \frac{1}{2}\right)$ | $\left(\frac{3}{2}, \frac{3}{2}\right)$ |
| 3 | $s_{e_2}$ | 1 | $-e_1 + e_2$ | $e_1$ | $\left(\frac{1}{2}, \frac{3}{2}\right)$ | $\left(\frac{1}{2}, \frac{3}{2}\right)$ |
| 4 | $s_{e_2}$ | 1 | $-e_1 + e_2$ | $e_1$ | $\left(\frac{1}{2}, \frac{3}{2}\right)$ | $\left(\frac{1}{2}, \frac{5}{2}\right)$ |

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# 14 Smaller blocks

In order to be able to assign representations to the smaller blocks, we must understand the automorphisms of an `atlas` block. These are symmetries coming from outer automorphisms (of the real group which are inner to the complex group) and disconnectedness of the real group $G(\mathbb{R})$. We define the notion of indistinguishability of block entries. For simplicity of statements, we assume in this section that $G(\mathbb{R})$ is semisimple.

**Definition 3** *Two block entries are said to be* ***weakly indistinguishable*** *if they are attached to the same twisted involution, and the nature of the simple roots agrees. Two block entries are* ***(strongly) indistinguishable*** *if all entries obtained by cross actions are pairwise indistinguishable. Two entries are (weakly or strongly)* $x$***-indistinguishable*** *if they are (weakly or strongly) indistinguishable, and the pairs of numbers parametrizing the corresponding pair* $(x, y)$ *coincide in the second number; i. e., they have the same* $y$ *parameter. Analogously, define weakly and strongly* $y$***-indistinguishable****.*

For example, lines 0 and 1 of the large block of $Sp(4, \mathbb{R})$ are strongly $x$-indistinguishable; lines 1 and 2 of the large block of $SO(2, 1)$ are strongly $y$-indistinguishable. Strongly $x$-indistinguishable pairs correspond to representations which differ by an outer automorphism of the group; strongly $y$-indistinguishable pairs to representations which differ by tensoring by a character $\chi$ which is trivial on the identity component of the group. Each of these maps gives an automorphism of the block output. For example, the large block output for $Sp(4, \mathbb{R})$ has one nontrivial automorphism, generated by switching 0 and 1, the large block of $SO(2, 1)$ has an automorphism generated by switching lines 1 and 2, and the large block of $G_2$ has no nontrivial automorphism. The block lines corresponding to each twisted involution may be partitioned into equal size sets of $x$-indistinguishable lines, and similarly for $y$-indistinguishable. The cardinality of these sets is constant for each Cartan subgroup. In order to assign representation characters, we must make a choice for each automorphism generator. In our $Sp(4, \mathbb{R})$ example, we chose #0 to be the line for the discrete series with Harish-Chandra parameter $(2, -1)$; all other characters were then uniquely determined. Assigning #1 to that particular discrete series results in a different, uniquely determined, assignment. In general, the choice corresponding to strong $x$-indistinguishability is given by a choice of starting point in the fundamental Cartan. These strongly $x$-indistinguishable pairs will collapse on a less compact Cartan depending on whether or not there is an element of the real Weyl group whose action agrees with the restriction of this automorphism to the Cartan.

The choice for strong $y$-indistinguishability happens on the dual side; in the large block, it is the choice of a basepoint, which determines the trivial representation. In smaller blocks, the real form of the dual group is different, so this choice has to be made in some other way. It is important to keep track of how many such choices have to be made. This is given by the size of the sets of $y$-indistinguishable lines for the most split Cartan subgroup. If $G(\mathbb{R})$ is connected, then there is no new choice to be made. If there is more than one possible fundamental basepoint to choose from, then we need to make a corresponding choice in the smaller blocks, as well. Strongly $y$-indistinguishable pairs collapse on Cartan subgroups on which the restriction of $\chi$ is trivial (for example, if $H(\mathbb{R})$ is connected).

Since the $x$-parameter determines $\theta$ (and hence a Cartan subgroup) and $\lambda$, and all possible values of $x$ appear in the large block, the only data to determine in the smaller blocks are the parameters $\kappa$. The restriction of $\Lambda$ to the connected component of the $\rho$-cover of $H(\mathbb{R})$ is uniquely determined by $\lambda$, so it only remains to understand what happens on the $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$ factors, so we need to look at the nature of the real roots.

**Proposition 4** *Let $H(\mathbb{R})$ and $\Lambda = \Lambda(\lambda, \kappa)$ be given, and let $\alpha$ be a real simple root. Then $\alpha$ is a parity root if and only if*

$$\langle \kappa - \lambda, \alpha^\vee \rangle$$

*is an even integer.*

Proposition 4 now goes a long way towards determining $\kappa$ in general; any remaining ambiguity comes from having to make the choice mentioned above.

The proposition also allows us to compute the size of the $y$-indistinguishable sets, by counting how many values of $\eta = \kappa - \lambda$ make all simple real roots parity roots.

**Proposition 5** *Given $\theta$ and $\lambda \in \rho + X^*(H)$, let $X^*(H)^{-\theta}$ be the elements of $X^*(H)$ fixed by $-\theta$, and $P$ the weight lattice. Then the cardinality of the $y$-indistinguishable sets for the corresponding Cartan subgroup $H(\mathbb{R})$ is the cardinality of the quotient*

$$2P \cap X^*(H)^{-\theta} / 2P \cap (1-\theta)X^*(H). \tag{110}$$

One difficulty is that outside the large block, these sets may be partitioned between different blocks.

We look at some examples to illustrate some of these facts.

## 14.1 $SL(4, \mathbb{R})$

The group $SL(4, \mathbb{R})$ is simply connected; in this case, the quotient (110) is always trivial, so there will be no choices to be made. We use this example to demonstrate how to use Proposition 4 to determine the $\Lambda$-data in the smaller blocks. It appears to be the smallest example for which the simpleminded method that we employed in Section 10 does not work for every representation.

There are three blocks, for the real forms $PSU(2,2)$, $PSU(3,1)$, and $PSU(4)$ of the dual group, respectively. We concentrate on the intermediate block $SL(4, \mathbb{R}) \times PSU(3,1)$, and compute only as much of the $\Lambda$-data for the large block as we need to get started.

There are three conjugacy classes of Cartan subgroups, two of which occur in the intermediate block. We choose our complex Cartan

$$H = \left\{ (z_1, z_2, z_3, z_4) \in \left(\mathbb{C}^\times\right)^4 : z_1 z_2 z_3 z_4 = 1 \right\}.$$

Then the fundamental Cartan involution $\theta_0$ and Cartan subgroup $H_0$ can be taken to be

$$\begin{aligned}\theta_0\left(z_1,z_2,z_3,z_4\right) &= \left(z_4^{-1},z_3^{-1},z_2^{-1},z_1^{-1}\right), \\ H_0 &= \left\{h_{r,\varphi,\psi}=\left(re^{i\varphi},r^{-1}e^{i\psi},r^{-1}e^{-i\psi},re^{-i\varphi}\right)\right\}\simeq\mathbb{C}^\times\times S^1.\end{aligned}$$

We display the first three lines of the large block:

```
0(0,11):  0  0  [C+,i1,C+]  3  1  3  (*,*)  (2,*)  (*,*)
1(1,11):  0  0  [C+,i1,C+]  4  0  4  (*,*)  (2,*)  (*,*)
2(2,10):  1  1  [C+,r1,C+]  6  2  5  (*,*)  (0,1)  (*,*)  2
```

Lines 0 and 1 are $x$-indistinguishable; this indistinguishability collapses on the other two Cartan subgroups. We choose #0 to be the representation with $\Lambda$-data

$$\begin{aligned}\lambda &= \left(\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2}\right)=\kappa, \\ \Lambda(h_{r,\varphi,\psi}) &= e^{i(3\varphi+\chi)}. \\ \alpha_1 &= e_1-e_2; \\ \alpha_2 &= e_2-e_3; \\ \alpha_3 &= e_3-e_4.\end{aligned}$$

We have to compose $\theta_0$ with $s_{e_2-e_3}$ to get

$$\begin{aligned}\theta_1\left(z_1,z_2,z_3,z_4\right) &= \left(z_4^{-1},z_2^{-1},z_3^{-1},z_1^{-1}\right), \\ H_1 &= \left\{h_{r,\varphi,x}=\left(re^{i\varphi},x,y,re^{-i\varphi}\right):re^{i\varphi}\in\mathbb{C}^\times,x\in\mathbb{R}^\times,y=x^{-1}r^{-2}\right\}\simeq\mathbb{C}^\times\times\mathbb{R}^\times.\end{aligned}$$

Then #2 has $\Lambda$-data

$$\begin{aligned}\lambda &= \left(\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2}\right)=\kappa, \\ \Lambda(h_{r,\varphi,x}) &= re^{3i\varphi}x; \\ \alpha_1 &= e_1-e_2; \\ \alpha_2 &= e_2-e_3; \\ \alpha_3 &= e_3-e_4.\end{aligned}$$

We are now ready to look at the intermediate block.

```
0(2,9):   1  1  [C+,rn,C+]  2  0  1  (*,*)  (*,*)  (*,*)  2
1(5,8):   2  1  [C+,C+,C-]  4  3  0  (*,*)  (*,*)  (*,*)  3,2,1
2(6,7):   2  1  [C-,C+,C+]  0  5  4  (*,*)  (*,*)  (*,*)  1,2,3
3(9,6):   3  1  [i2,C-,rn]  3  1  3  (7,9)  (*,*)  (*,*)  2,1,3,2,1
4(10,5):  3  1  [C-,i2,C-]  1  4  2  (*,*)  (6,7)  (*,*)  1,2,3,2,1
5(11,4):  3  1  [rn,C-,i2]  5  2  5  (*,*)  (*,*)  (6,8)  1,2,1,3,2
6(12,0):  4  2  [rn,r2,r2]  6  7  8  (*,*)  (4,*)  (5,*)  1,2,1,3,2
7(12,1):  4  2  [r2,r2,rn]  9  6  7  (3,*)  (4,*)  (*,*)  1,2,1,3,2
8(12,2):  4  2  [rn,rn,r2]  8  8  6  (*,*)  (*,*)  (5,*)  1,2,1,3,2
9(12,3):  4  2  [r2,rn,rn]  7  9  9  (3,*)  (*,*)  (*,*)  1,2,1,3,2
```

The first six lines are attached to the intermediate Cartan $H_1$; we can obtain the $\Lambda$-data for #0 by changing the $\kappa$-value for those of #2 in the large block so that $\alpha_2$ becomes a nonparity root. Consider $\eta = \kappa - \lambda$ instead. Since

$$\kappa + \theta_1\kappa = \lambda + \theta_1\lambda,$$

$\eta$ must satisfy

$$\theta_1\eta = -\eta, \tag{111}$$

and we must also have $\eta \in X^*(H) = \{(a,b,c,d) \in \mathbb{R}^4 : a-b, b-c, c-d \in \mathbb{Z}\}$ (modulo $\mathbb{R}$, diagonally embedded). Condition (111) says that if $\eta = (a,b,c,d)$ then $a = d$. In order to turn $\alpha_2 = e_2 - e_3$ into a nonparity root, we can choose

$$\eta = (0,0,1,0)\,,$$

so we have

$$\begin{aligned}
\lambda &= \left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right), \\
\kappa &= \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right) \\
\Lambda(h_{r,\varphi,x}) &= re^{3i\varphi}\,|x|\,; \\
\alpha_1 &= e_1 - e_2; \\
\alpha_2 &= e_2 - e_3; \\
\alpha_3 &= e_3 - e_4.
\end{aligned}$$

The other five sets of $\Lambda$-data attached to this Cartan may now be obtained by cross actions as explained earlier. Recall that $\eta = \kappa - \lambda$ is not affected by imaginary or complex cross actions, so we obtain

| # | CSG | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\lambda$ | $\kappa$ | $\eta$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | $e_1 - e_2$ | $e_2 - e_3$ | $e_3 - e_4$ | $\left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$ | $\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right)$ | $(0,0,1,0)$ |
| 1 | 1 | $e_1 - e_2$ | $e_2 - e_4$ | $-e_3 + e_4$ | $\left(\frac{3}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$ | $\left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ | $(0,0,1,0)$ |
| 2 | 1 | $-e_1 + e_2$ | $e_1 - e_3$ | $e_3 - e_4$ | $\left(\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$ | $\left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{3}{2}\right)$ | $(0,0,1,0)$ |
| 3 | 1 | $e_1 - e_4$ | $-e_2 + e_4$ | $e_2 - e_3$ | $\left(\frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}, +\frac{1}{2}\right)$ | $\left(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right)$ | $(0,0,1,0)$ |
| 4 | 1 | $-e_1 + e_2$ | $e_1 - e_4$ | $-e_3 + e_4$ | $\left(\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$ | $\left(\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ | $(0,0,1,0)$ |
| 5 | 1 | $e_2 - e_3$ | $-e_1 + e_3$ | $e_1 - e_4$ | $\left(-\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{3}{2}\right)$ | $\left(-\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right)$ | $(0,0,1,0)$ |

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To get to Cartan #2, we must perform a Cayley transform; all such Cayley transforms are double-valued, so we need to use the parity condition to distinguish between the two images. The Cartan $H_2$ is the split Cartan, so $\theta_2$ is inversion, and

$$H_2 = \left\{ h_{x_1,x_2,x_3,x_4} = (x_1, x_2, x_3, x_4) \in \left(\mathbb{R}^\times\right)^4 : x_1 x_2 x_3 x_4 = 1 \right\}.$$

We start with #4 and perform a Cayley transform through $\alpha_2 = e_1 - e_4$. The two images will correspond to #6 and #7. They differ by cross action through $\alpha_2 = e_1 - e_4$, so their parameters $\kappa$ differ by $\alpha_2 = e_1 - e_4$; this means that one of them has $\eta = (0,0,1,0)$, the other $\eta = (1,0,1,-1)$ (which gives the same character as $(0,1,0,0)$ since the difference is in $(1-\theta_2)X^*(H)$). In #6, $\alpha_1 = -e_1 + e_2$ is a nonparity root, and

$$\begin{aligned} \langle (0,0,1,0), \alpha_1 \rangle &= 0 \text{ (even)} \\ \langle (0,1,0,0), \alpha_1 \rangle &= 1 \text{ (odd)}, \end{aligned}$$

so $\eta = (0,1,0,0)$ corresponds to #6. The remaining parameters may now be obtained by cross actions through real roots (or you can check the parity conditions), so we obtain the following table of $\Lambda$-data:

| # | CSG | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\lambda$ | $\kappa$ | $\eta$ |
|---|---|---|---|---|---|---|---|
| 6 | 2 | $-e_1 + e_2$ | $e_1 - e_4$ | $-e_3 + e_4$ | $\left(\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$ | $\left(\frac{1}{2}, \frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$ | $(0,1,0,0)$ |
| 7 | 2 | $-e_1 + e_2$ | $e_1 - e_4$ | $-e_3 + e_4$ | $\left(\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$ | $\left(\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ | $(0,0,1,0)$ |
| 8 | 2 | $-e_1 + e_2$ | $e_1 - e_4$ | $-e_3 + e_4$ | $\left(\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$ | $\left(\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$ | $(1,0,0,0)$ |
| 9 | 2 | $-e_1 + e_2$ | $e_1 - e_4$ | $-e_3 + e_4$ | $\left(\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$ | $\left(\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{1}{2}\right)$ | $(0,0,0,1)$ |

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## 14.2 The compact block

Let $G(\mathbb{R})$ be split, and consider a block given by a compact real form of the dual group. Assume that $\rho \in X^*(H)$, so that this block corresponds to the infinitesimal character in the translation family of $\rho$. (This is not a necessary condition; consider, e. g., $G(\mathbb{R}) = PSO(7,7)$.) The block consists of only one line, corresponding to a principal series representation with all roots real nonparity roots. Since $\lambda = \rho$ and $\kappa \in \rho + X^*(H)$, we have $\eta = \kappa - \lambda \in X^*(H)$. One option for $\eta$ is $\eta = \rho$ since

$$\langle \rho, \alpha^\vee \rangle = 1$$

for all simple roots $\alpha$, so this makes all simple roots nonparity roots. In general, this is not the only choice; the $y$-indistinguishable sets for the split Cartan may have more than one element; in this case, there should be the same number of representations. This is where strong real forms, rather than real forms come in. The `block` and `kgb` output for different strong real forms (corresponding to the same weak real form) look the same, so `atlas` makes a choice and provides it only once. We can check how many strong real forms there are by looking at the `strongreal` output for the dual group, or by computing the quotient (110).

Consider the example $G(\mathbb{R}) = PSO(4,4)$. A look at the large block tells us that the $y$-indistiguishable sets for the split Cartan have four elements. The compact form of the dual group is $Spin(8)$; using the `strongreal` command we can see that there are four strong real forms, each of which is in the class of the split real form, hence corresponds to the infinitesimal character $\rho$:

```
empty:type
Lie type:  D4 sc e
main:strongreal
(weak) real forms are:
0:  so(8)
1:  so(6,2)
2:  so*(8)[0,1]
3:  so*(8)[1,0]
4:  so(4,4)
enter your choice:  0
there is a unique conjugacy class of Cartan subgroups
Name an output file (return for stdout, ?  to abandon):

there are 4 real form classes:
class #0:
real form #4:  [0,1,2,4,5,6,8,9,10,12,13,14] (12)
real form #0:  [3] (1)
real form #0:  [7] (1)
real form #0:  [11] (1)
real form #0:  [15] (1)
```

```
class #1:
real form #2:  [0,1,2,7,8,9,10,15] (8)
real form #2:  [3,4,5,6,11,12,13,14] (8)
class #2:
real form #3:  [0,2,3,4,6,7,9,13] (8)
real form #3:  [1,5,8,10,11,12,14,15] (8)
class #3:
real form #1:  [0,2,3,5,9,12,14,15] (8)
real form #1:  [1,4,6,7,8,10,11,13] (8)
```

If we write the roots in the standard way,

$$\Psi = \{\pm e_i \pm e_j : 1 \leq i < j \leq 4\},$$

then the root lattice is

$$R = \left\{(a,b,c,d) \in \mathbb{Z}^4 : a+b+c+d \in 2\mathbb{Z}\right\} = X^*(H),$$

and

$$\rho = (3,2,1,0) \in X^*(H).$$

The parameter $\eta$ is defined up to

$$(1-\theta)\, X^*(H) = 2X^*(H),$$

so the choices for $\eta$ are given by representatives of $X^*(H)/2X^*(H)$; we can choose

$$\begin{aligned} \eta_1 &= (3,2,1,0) \\ \eta_2 &= (3,2,-1,0) \\ \eta_3 &= (4,3,2,1) \\ \eta_4 &= (4,3,2,-1) \end{aligned}$$

to get four inequivalent representations, one for each of the four compact blocks.

Let's check the number of set elements using the quotient (110). For the split Cartan, we have $\theta = -1$,so

$$\begin{aligned} X^*(H)^{-\theta} &= X^*(H), \\ P &= \mathbb{Z}^4 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^4, \\ 2P &= (2\mathbb{Z})^4 \cup (2\mathbb{Z}+1)^4 = 2P \cap X^*(H)^{-\theta}, \\ 2P \cap (1-\theta)\, X^*(H) &= (1-\theta)\, X^*(H) = 2X^*(H). \end{aligned}$$

The quotient $2P/2X^*(H)$ has indeed four elements; representatives may be taken as

$$(0,0,0,0)\,, (1,1,1,1)\,, (2,0,0,0), (3,1,1,1).$$

## 14.3 Weakly indistinguishable entries

It can happen that a double-valued Cayley transform leads to weakly (but not strongly) $y$-indistinguishable parameters. In this case the choice of representations is uniquely determined. In the large block, this is not a problem because of our trick using basepoints. In general, we need to determine which representation is which. The easiest way is to avoid this Cayley transform, and find a different path to one of the parameters which does not involve any weakly indistinguishable double-valued Cayley transform, and then obtain the other parameter through the appropriate real cross action. This may be always possible (is this true?). If all else fails, we can guess an assignment for the two parameters, then perform cross actions to distinguishable parameters, and check whether our assignment was the correct one, by checking the nature of the roots.

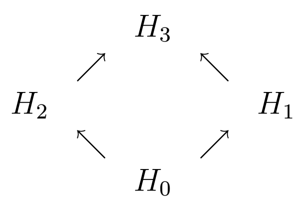
For example, consider the block $Sp(8, \mathbb{R}) \times SO(6, 3)$. Entries 120 and 121 may be obtained from 115 by a double-valued Cayley transform through the first root. The two entries are weakly $y$-indistinguishable. We can obtain the $\Lambda$-data for these two parameters from those of 115 by writing down the two possible parameters obtained by Cayley transform (their $\kappa$'s differ by $\alpha_1$), then perform complex cross actions through $\alpha_3$ to get the parameters for entries 132 and 133. These entries are not indistinguishable; they differ by the nature of the real root $\alpha_2$. Using Proposition 4, we can match up the parameters for 132 and 133, and thus those for 120 and 121. This kind of procedure would be necessary if every Cayley transform (increasing length) leading to Cartan #6 were double-valued. An easier way would be to obtain the parameters of, say, #102 from #91, by Cayley transform through $\alpha_4$, and then perform cross actions to obtain the $\Lambda$-data for all other lines attached to Cartan #6.

## 14.4 Dependent choices

If $G(\mathbb{R})$ is disconnected so that a choice has to be made, it is important that this choice is made in a consistent fashion. In our walk through the block, assigning parameters to block lines, we may have to make a choice at more than one double-valued Cayley transform with indistinguishable images; these choices are not necessarily independent. The assignment of representations must be independent of the path. If the $y$-indistiguishable sets associated with the most split Cartan have cardinality two, then only one choice may be made freely; all others depend on this first choice. It is helpful to choose the path through the block in a clever way so that we encounter a minimal number of double-valued indistinguishable Cayley splits. Consider, for example, $SO(3, 2)$ which has two connected components, and a nontrivial character, $sgn$, which is trivial on the identity component. The group has four (conjugacy classes of) Cartan subgroups:

$$\begin{array}{rcl} H_0 & \simeq & \left(S^1\right)^2 \\ H_1 & \simeq & S^1 \times \mathbb{R}^\times \\ H_2 & \simeq & \mathbb{C}^\times \\ H_3 & \simeq & \left(\mathbb{R}^\times\right)^2 \end{array}$$

The restriction of $sgn$ to $H_0$ and $H_2$ is trivial (since they are connected), and its restriction to the other two is nontrivial. There are Cayley transforms as indicated in the diagram:




The Cayley transforms from $H_0$ to $H_1$ and from $H_2$ to $H_3$ will be double-valued with indistinguishable images. The choices to be made will be dependent on each other. One can avoid this difficulty by choosing the path from $H_1$ to $H_3$ instead, so that only one choice occurs, with the Cayley transform from $H_0$ to $H_1$.

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